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A CLASS OF RIEMANNIAN HOMOGENEOUS SPACES

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In M. Berger's classification [1] of normal Riemannian homogeneous spaces of strictly positive curvature, appear various classes of Riemannian homogeneous metrics which can be put on odd-dimensional spheres — but which do not have constant curvature. In this paper, we investigate one of these classes, viz. $\{SU(n+1) \times R/SU(n) \times R\}_\alpha \equiv {}_{def} M_\alpha^n$ (cf. details below). In particular, we (i) calculate the conjugate locus (in the tangent space) of any point, (ii) calculate the totally geodesic submanifolds of constant curvature, and (iii) consider closed geodesics in M_α^n and their relationship to a lemma of W. Klingenberg [5, Theorem 1].

§ 1 concerns itself with the explicit construction of M_α^n, and § 2 is devoted to (i). The key tool of § 2 is the writing of Jacobi's equations of geodesic deviation in the canonical connection of G/H. (This connection is not the Levi-Civita connection, but nevertheless has the same geodesics.) For the necessary background, the reader is referred to [3]. In § 3 we dispose of (ii), in § 4 we calculate the pinching of M_α^n, and in § 5 we discuss (iii). § 5 is a direct generalization of M. Berger's argument in [2, pp. 9–12], and § 6 consists of remarks relating the spaces M_α^n to problems in Riemannian geometry.

1. The space M_α^n

We first let E_{jk} denote the matrix, whose r-th row and s-th column are given by $\delta_{jr}\delta_{ks}$, i.e., E_{jk} has a 1 in the j-th row and k-th column and zeros elsewhere, and we set

$$\begin{aligned} A_{jk} &= \sqrt{-1}\,(E_{jj} - E_{kk}) \,, \\ B_{jk} &= \phantom{\sqrt{-1}}\,(E_{jk} - E_{kj}) \,, \\ C_{jk} &= \sqrt{-1}\,(E_{jk} + E_{kj}) \,. \end{aligned}$$

Then a basis of $\mathfrak{a}_n =$ Lie algebra of $SU(n+1)$ is given by $\{A_{l,l+1}\colon l = 1, \cdots, n;\ B_{rj}, C_{rj}\colon 1 \le r < j \le n+1\}$. For a bi-invariant metric on $\mathfrak{a}_n$ we choose

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{trace} XY \,.$$

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