J. DIFFERENTIAL GEOMETRY
4 (1970) 1-4

POSITIVELY CURVED n-MANIFOLDS IN R^{n+2}

ALAN WEINSTEIN

Introduction

In view of the difficulty of classifying *all* compact Riemannian manifolds with strictly positive sectional curvature, we make the additional hypothesis that the manifold is isometrically immersed in a Euclidean space with codimension 2. In § 1 we prove a theorem in what B. O'Neill has called "pointwise differential geometry" (i.e. linear algebra). This theorem is applied in § 2 to obtain results about the manifolds specified in the title. For instance, we show that a metric of positive curvature on $S^2 \times S^2$ cannot be induced by an immersion in R^6.

1. An algebraic theorem

Let V and W be real vector spaces of finite dimensions n and p respectively, and $B: V \times V \to W$ a symmetric bilinear form on V with values in W. Suppose $n \geq 2$ and W has an inner product $\langle\ ,\ \rangle$. Define the *associated curvature form* $R_B: \Lambda^2 V \times \Lambda^2 V \to R$ by

$$R_B(x \wedge y, z \wedge w) = \langle B(x,z), B(y,w)\rangle - \langle B(x,w), B(z,y)\rangle \ .$$

R_B is again symmetric, and is positive definite iff $R_B(\omega,\omega) > 0$ whenever $\omega \neq 0$. We say that R_B has *positive sectional values* iff $R_B(x \wedge y, x \wedge y) > 0$ whenever $x \wedge y \neq 0$.

Consider the following conditions on B:

(a) There exists an orthonormal basis $\{e_1, \cdots, e_p\}$ for W such that the real-valued forms on V defined by $(x,y) \mapsto \langle B(x,y), e_i\rangle$ are all positive definite.

(b) R_B is positive definite.

(c) R_B has positive sectional values.

Theorem 1. (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *If* $p = 2$, *then* (c) $\Rightarrow$ (a). *In fact, let* $p = 2$ *and* $\mathscr{P} = \{B \mid R_B$ *has positive sectional values*$\}$. *Then there are continuous functions* e_1 *and* e_2 *from* $\mathscr{P}$ *to* W, *canonically determined by an orientation of* W, *such that for each* $B \in \mathscr{P}$, $\{e_1(B), e_2(B)\}$ *is an orthonormal frame for* W, *and the forms* $(x,y) \mapsto \langle B(x,y), e_i(B)\rangle$ *are both positive definite.*

Communicated by W. P. A. Klingenberg, January 17, 1969.