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SIGN CHANGES, EXTREMA, AND CURVES OF MINIMAL ORDER

H. GUGGENHEIMER

Let $\{p_k(t)\}$ be a set of polynomials with real coefficients indexed by the degree. If the moments of a continuous function $f(t)$ for these polynomials vanish, that is, if

$$\int_0^1 f(t)p_k(t)dt = 0 , \qquad 0 \leq k \leq n ,$$

and if $f(t)$ does not vanish identically, then $f(t)$ has at least $n + 1$ changes of sign in $(0, 1)$. A similar result holds for trigonometric integrals, namely, if

$$\int_0^1 f(t) \cos 2\pi\, kt\, dt = \int_0^1 f(t) \sin 2\pi\, kt\, dt = 0 , \qquad 0 \leq k \leq n ,$$

then $f(t)$ has at least $2n + 1$ sign changes in $(0, 1)$ unless it vanishes identically. The theorems and the technique of proof are explained in Pólya-Szegö [1, p. 65, Problems 140 et seq.].

The theorem about the connection between the vanishing of the Fourier coefficients and the number of sign changes of a function has many important applications in geometry. A survey of these applications is given in [2].

In this note we give a general theorem based on an order property of curves in n-space, and during the course of this investigation we are led to an interesting unsolved problem about curves in n-space. We also give, as applications, some new vertex theorems in differential geometry and differential equations, and obtain a much shortened proof and a generalization of a theorem about intermediate values [2, Theorem 3].

1. We say that a continuous curve in Cartesian n-space

$$\boldsymbol{u}\colon [0, 1] \to \boldsymbol{R}^n$$

is of *minimal order* if no $(n - 1)$-plane intersects $\{\boldsymbol{u}(t)\}$ in more than n points, or in Haupt's terminology, the set $\{\boldsymbol{u}([0, 1])\}$ is a continuum of POW n for the $(n - 1)$-planes in n-space. Hence [3, §5.2.4, 2. Satz], $\boldsymbol{u}$ is a simple arc or a simple closed curve. In fact, the existence of a multiple point can be

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