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PERIODIC GEODESICS ON COMPACT RIEMANNIAN MANIFOLDS

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The interest in periodic geodesics arose at a very early stage of differential geometry, and has grown rapidly since then. It is a basic general problem to estimate the number of distinct periodic geodesics $c\colon \boldsymbol{R} \to M$, $c(t+1) = c(t)$, on a complete riemannian manifold M in terms of topological invariants. Here periodic geodesics are always understood to be non-constant, and two such curves c_1, c_2 will be said to be distinct if they are geometrically different, $c_1(\boldsymbol{R}) \neq c_2(\boldsymbol{R})$.

If M is non-compact, then it is even difficult to find reasonable conditions for the existence of at least one periodic geodesic, see § 4. However, in the compact case many results are available. It is classical and rather elementary that any non-trivial conjugacy class of the fundamental group $\pi_1(M)$ gives rise to periodic geodesics, and very often the existence of a larger number of distinct periodic geodesics can be deduced from further properties of $\pi_1(M)$; compare [4, p. 240], [7], and also § 4. When M is simply connected, the problem is getting much more delicate. Here the first result was obtained in 1905 by Poincaré, who proved that there exists a periodic geodesic on every surface analytically equivalent to the euclidean sphere S^2. Yet rather late, in 1952, Fet and Lusternik established the theorem that on any compact riemannian manifold M at least one geodesic is periodic. Several authors have proved the existence of certain finite numbers of distinct periodic geodesics for special topological types of manifolds, partly under restrictive metric conditions. We mention the work of Lusternik, Schnirelmann, Morse, Fet, Alber, and Klingenberg; for references see [11].

In § 4 of this paper we shall prove: There always exist infinitely many distinct periodic geodesics on an arbitrary compact manifold, provided some weak topological condition holds. In fact, it seems that our condition will be satisfied except in comparatively few cases. Until now it was not even possible to decide whether there is some compact simply connected differentiable manifold M with infinitely many distinct periodic geodesics for all riemannian structures on M.

As yet the most natural and successful way of dealing with periodic geodesics

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