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DISCRETE NILPOTENT SUBGROUPS OF LIE GROUPS

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1. Introduction

C. L. Siegel [5] has shown that the area of the fundamental domain of a totally discontinuous group of motions of the hyperbolic plane is at least $\pi/21$. Recently D. A. Kazhdan and G. A. Margulis [4] proved that every semisimple Lie group without compact factor has a neighborhood U of the identity e such that, given any discrete subgroup Γ of G, there exists $g \in G$ with the property that $g\Gamma g^{-1} \cap U = \{e\}$. This implies that the volume of the fundamental domain of discrete subgroups of G (when considered as a group of left translations of G, or as the group of isometries on the symmetric space associated with G) has a positive lower bound. It is the aim of this paper to give a *quantitative* study of the neighborhood U. Two properties of discrete nilpotent subgroups of Lie groups will be established; they lead directly to an estimate of the size of U. One of the properties is a sharpening of a theorem of Zassenhaus [8]. We note that, whereas Kazhdan-Margulis used results on algebraic groups, our proof here consists in some elementary geometrical arguments.

Let G be a semisimple Lie group, $\mathfrak{G}$ its Lie algebra, k the Killing form over $\mathfrak{G}$, and $\sigma: \mathfrak{G} \to \mathfrak{G}$ a Cartan involution. Define an inner product $\langle\,\rangle$ by putting $\langle X, Y\rangle = -k(X, \sigma Y), X, Y \in \mathfrak{G}$; it gives a left invariant Riemannian metric, and hence a distance function ρ, over the group space G. This distance function ρ is not unique, but any two of such differ only by an inner automorphism of G. With the semisimple Lie algebra $\mathfrak{G}$, we associate a positive real number R_G which can be computed from the root system. For example, $R_{SL(n,R)} = c\sqrt{n}$, $R_{SU(p,q)} = c(p+q)^{1/2}$ where c is approximately $277/1000$. Using these notations, we can describe our main results as follows:

I. *For every discrete subgroup Γ of a semisimple Lie group G, the set $\{g \in \Gamma: \rho(e, g) \leq R_G\}$ generates a nilpotent subgroup.*

II. *Suppose G to be a semisimple Lie group without compact factor. Let $\mathfrak{G}_\pi$ be the totality of elements X in the Lie algebra of G such that all the eigenvalues of $\operatorname{ad} X$ have their imaginary parts lying in the open interval $(-\pi, \pi)$, and $G_\pi = \{\exp X: X \in \mathfrak{G}_\pi\}$. Then, given any nilpotent discrete subgroup Γ of G and any compact neighborhood C of e with $C \subset G_\pi$, there exists $g \in G$ such that $g\Gamma g^{-1} \cap C = \{e\}$.*

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