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HOLOMORPHIC VECTOR FIELDS AND THE FIRST CHERN CLASS OF A HODGE MANIFOLD

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In a recent paper [2] Bott has proved that if a connected compact complex manifold admits a nonvanishing holomorphic vector field, then all the Chern numbers of M vanish.

In this paper we first prove the following theorem.

Theorem 1. *Let M be a connected Hodge manifold, and suppose that there exists a nonvanishing holomorphic vector field X in M. Then there exists a nonvanishing holomorphic 1-form ω in M such that $\omega(X) \neq 0$. In particular, the first Betti number $b_1(M)$ of M is different from zero.*

We shall then study the structure of a Hodge manifold with zero first Chern class. We denote by $c_1(M)$ and $q(M)$ the first Chern class and the irregularity (i.e., one half of the first Betti number) of M respectively, and by G the identity component of the group of all holomorphic transformations of M. The group G is a connected complex Lie group.

We shall prove the following two theorems which sharpen some of the recent results of Lichnerowicz [5].

Theorem 2. *Let M be a connected Hodge manifold such that $c_1(M) = 0$. Then the group G is an abelian variety of dimension $q(M)$ and the isotropy subgroup of G at any point in M is a finite group.*

Theorem 3. *Let M be a connected Hodge manifold and assume that $c_1(M) = 0$ and $q(M) > 0$. Then there exist an abelian variety A and a connected Hodge manifold F with the following properties.*

a) *$c_1(F) = 0$ and $q(F) = 0$;*

b) *$A \times F$ is a finite regular covering space of M and the group of covering transformations is solvable.*

After having finished this work, the author learned that Calabi stated these two theorems in his paper [4] as his well-known conjecture, and proved them under the assumption that M is a connected compact Kähler manifold with vanishing Ricci curvature tensor.

1. Let M be a connected compact Kähler manifold, and $\mathfrak{h}$ and $\mathfrak{g}$ denote, respectively, the complex vector space of all holomorphic 1-forms and the complex Lie algebra of all holomorphic vector fields in M. Then $\dim \mathfrak{h} = q(M)$ and we can identify $\mathfrak{g}$ with the Lie algebra of the group G. If $\omega \in \mathfrak{h}$ and $X \in \mathfrak{g}$, then

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