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DIVERGENCE-PRESERVING GEODESIC SYMMETRIES

J. E. D'ATRI & H. K. NICKERSON

On a differentiable manifold of dimension n, a local volume element ω (non-vanishing n-form) determines the divergence, relative to ω, of a local vector field Y by

$$(\operatorname{div} Y)\omega = L_Y\omega . \tag{1}$$

On a Riemannian manifold, there are two canonical choices of ω on any simply-connected neighborhood, differing only in sign, corresponding to the two possible orientations: ω is defined by the condition that any positively-oriented orthonormal frame shall span unit volume. By (1), either of these choices determines the same local map $Y \to \operatorname{div} Y$.

In this paper, we consider Riemannian spaces in which every local geodesic symmetry is divergence-preserving (or, equivalently, volume-preserving up to sign). This class of spaces obviously includes the Riemannian locally symmetric spaces, in which every local geodesic symmetry satisfies the stronger condition of being an isometry. It also includes the harmonic spaces with positive-definite metric. An example is given, which shows that our class is strictly larger than either of these subclasses.

We derive an infinite sequence of necessary conditions on the curvature (sufficient in the case of an analytic manifold), which are a subset of the necessary conditions for a harmonic space.

1. Equivalent statements of the property

All Riemannian structures, functions, vector fields, mappings, etc., will be assumed differentiable of class C^∞. In so far as possible, the notation will follow [2]. The notations Λ, Ω, and Π are chosen to agree with [4].

Lemma 1.1. *A local diffeomorphism is divergence-preserving if and only if it preserves volume to within a constant factor.*

Proof. Let $F\colon M \to N$ be a local diffeomorphism, and define the non-zero local scalar function h on M by $F^*\tilde{\omega} = h\omega$, where ω, $\tilde{\omega}$ are given local volume elements on M, N respectively. If Y is a local vector field on M, then

$$(\operatorname{div} F_*Y) \circ F = \operatorname{div} Y + (Y \cdot h)/h .$$

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