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HIGHER ORDER CONSERVATION LAWS

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1. Introduction

Let A denote the ring of germs of analytic functions at a point of an analytic manifold, and $\mathscr{E}$ the localization of the A-module of differential forms on this manifold. Let $\underline{h} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$. Then $\theta \in \mathscr{E}$ is said to be a conservation law for $\underline{h}$ if both θ and $\underline{h}\theta$ are exact. More generally, θ is a conservation law for a finite collection of elements $\underline{h}_i \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ if θ and $\underline{h}_i\theta$ are all exact. The problem which has been investigated in previous papers (see [4], [5], and [6]) has been that of determining all conservation laws $\theta \in \mathscr{E}$ for a given finite collection of endomorphisms $\underline{h}_i$ of $\mathscr{E}$. Certain conditions have usually been imposed on the $\underline{h}_i$, namely that the $\underline{h}_i$ have distinct eigenvalues and that certain concomitants associated with the $\underline{h}_i$ vanish identically.

The problem which is studied in this paper generalizes the conservation law problem in another direction. Let $\Lambda^p\mathscr{E}$ denote the differential forms of degree p, and $T \in \operatorname{Hom}_A(\Lambda^p\mathscr{E}, \Lambda^p\mathscr{E})$. Then $\theta \in \Lambda^p\mathscr{E}$ is called a conservation law of order p for T, if both θ and $T\theta$ are exact. The problem is then one of determining all conservation laws for a given T. In the sequel the case in which elements T are induced by an element $\underline{h} \in \operatorname{Hom}_A(\mathscr{E}, \mathscr{E})$ is considered, and the main result is stated in Theorem 3.4. An alternate characterization of conservation laws is then given in § 4.

2. Preliminaries

Let $(u^1, \cdots, u^n)$ form a local coordinate system in A at some point P of the manifold at which $u^i(P) = 0$. If d is the operation of exterior differentiation, then $(du^1, \cdots, du^n)$ is locally a basis for $\mathscr{E}$ which has dimension n. Moreover, $\mathscr{E}$ is a free A-module, and the exterior algebra $\Lambda^*\mathscr{E}$ generated by $\mathscr{E}$ is a direct sum

$$\Lambda^*\mathscr{E} = \Lambda^0\mathscr{E} \oplus \Lambda^1\mathscr{E} \oplus \cdots \oplus \Lambda^n\mathscr{E} ,$$

where $\Lambda^0\mathscr{E} = A$ and $\Lambda^1\mathscr{E} = \mathscr{E}$. The dimension of $\Lambda^p\mathscr{E}$, the space of differential

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