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CRITICAL POINTS OF THE DISPLACEMENT FUNCTION OF AN ISOMETRY

VILNIS OZOLS

Introduction

Given a Riemannian manifold M and a group of isometries of M it is natural to study the fixed point set of this group. This problem was considered by S. Kobayashi in [9], [10], and by R. Bott in [2], in the case where the group is a 1-parameter group of isometries. In [4], Kobayashi shows that if $\{g_t\}$ is such a group, then the fixed point set of $\{g_t\}$ is a totally geodesic submanifold of even codimension. In fact, his proof shows that the fixed point set of any group of isometries is a totally geodesic submanifold. The fixed point set of the 1-parameter group $\{g_t\}$ is just the set of zeros of the associated Killing vector field X, and in [7] and [8] R. Hermann considers the more general problem of the critical points of the function $|X|^2$ giving the square of the length of X. He shows that these critical points are exactly the points lying on geodesic orbits of $\{g_t\}$. Moreover, he shows that if M has curvature $K \leq 0$, then the set of critical points of $|X^2|$ is convex (that is, any geodesic segment between two critical points lies in the critical set).

We consider the still more general situation of a single isometry f, and look at the critical point set Crit (f) of the function δ_f^2, where $\delta_f(x) =$ distance $(x, f(x))$. It is evident that Crit (f) contains the fixed points of f.

In Chapter I we let M be any Riemannian manifold and $f: M \to M$ an isometry whose displacement δ_f is small enough so that f takes each point into the complement of its cut locus. We say such an isometry has "small displacement." The main theorems are:

(1.2.1) Theorem. *Let $f: M \to M$ be an isometry of small displacement and $x \in M$. Then $x \in$ Crit (f) if and only if f preserves the unique minimizing geodesic between x and $f(x)$.*

(1.3.4) Theorem. *Let M have curvature $K \leq 0$, and assume $f: M \to M$ is an isometry of small displacement. Then*

(i) Crit (f) *is a totally geodesic submanifold possibly with boundary,*

(ii) δ_f *takes its absolute minimum on* Crit (f).

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