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THE TOPOLOGY OF TAUT RIEMANNIAN MANIFOLDS WITH POSITIVE RISEC PINCHING

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0. Introduction

Suppose N is a smooth compact Riemannian manifold of dimension n. Geometers have long known that innocuous-appearing restrictions on the set $K(N)$ of sectional curvatures of N can lead to strong implications for the topology of N. For example, if $n = 2$ and $K(N) > 0$, then the classical Gauss-Bonnet theorem [9] leads to the conclusion that N is homeomorphic to the 2-sphere S^2 or the real projective plane P^2. Bishop and Goldberg [2] show how to derive further information from the generalized Gauss-Bonnet formula in higher dimensions. Again, if n is even, N is orientable, and $K(N) > 0$, then the fundamental group $\pi_1(N)$ is trivial [14]. A most remarkable series of advances have recently culminated in the Sphere Theorem [5]: if $K(N) \subset (k, 1]$ where $k > 1/4$, and $\pi_1(N)$ is trivial, then N is homeomorphic to the n-sphere S^n.

Berger and Bott [1] have studied the sums of the Betti numbers (over an arbitrary coefficient field) of the loop space of N, assuming that the sectional curvatures are k-pinched. They show that the sum of the Betti numbers of dimension at most λ is majorized by an exponential-polynomial expression in λ. Under certain conditions causing the vanishing of lower-order Betti numbers of N, it is possible to derive estimates for sums of the Betti numbers of N itself.

We consider here the possibility that the sectional curvatures of N can take on negative values, but with the proviso that the Ricci curvatures are always positive. This leads us to replace the concept of pinching by two new numerical characters of the metric, namely, the *tautness* and *risec pinching*; these numbers are defined precisely in § 1. It is then possible to alter the method of [1] to get new estimates for sums of Betti numbers, given below in Theorem 5.3. We apply this estimate to examine γ-holomorphically pinched Kähler manifolds M in the case where $c_d < \gamma < 2/3$ (c_d is some constant depending on the complex dimension d of M). In this range for γ, M will have positive Ricci curvatures and may have negative sectional curvatures. Known information about γ-holomorphically pinched Kähler manifolds is scarce and typically restricted to the case when γ is closer to 1 (e.g. $\gamma > 4/5$ in [3]).

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