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SOME DECOMPOSITIONS OF THE SPACE OF SYMMETRIC TENSORS ON A RIEMANNIAN MANIFOLD

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1. Introduction

In this article we consider a *compact* C^∞ manifold M and endow it with a riemannian structure g. For such a riemannian manifold (M, g), the space A^p of exterior differential forms carries an elliptic operator and the de Rham laplacian Δ, and has an orthogonal decomposition

$$(1.1) \qquad A^p = \ker\Delta \oplus dA^{p-1} \oplus \delta A^{p+1} ,$$

orthogonal with respect to the global scalar product on A^p. Moreover the dimension of $\ker\Delta$ is equal to the p-th Betti number of M, thanks to de Rham's theorem.

The simplest space of tensors to consider, besides the A^p's, is the space S^2 of symmetric bilinear differential forms on M. It is natural to look for a decomposition of S^2 like (1, 1). In this article we give all the reasonable decompositions of S^2, which we are aware of. Unhappily we have no essential applications of them, because of the lack of some kind of a de Rham theorem, connecting topological invariants of M with the dimension of the kernel of the elliptic operators considered on S^2. However we think it is worthwhile to list and prove these decompositions, hoping the reader will find interest in the problems and questions which naturally arise.

After fixing notations in §2 we give in §§3 and 4 the decomposition (3.1), which essentially yields the subspace $\delta^{-1}(0)$ of S^2 as the tangent space at g of the space $\mathcal{M}/\mathcal{D}$ of classes of riemannian structures on M under diffeomorphisms. In §5 we give two decompositions for manifolds of constant scalar curvature which are naturally associated to deformations of the scalar curvature. One of these is due to L. Nirenberg. §6 lists four elliptic operators on S^2; in §6.c an application is made to minimal surfaces; Corollary 6.2 was suggested to us by J. Simons. §§7 and 8 are concerned with Einstein manifolds. As an application of the decomposition (3.1) of §3, we find that the space of Einstein

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