J. DIFFERENTIAL GEOMETRY
3 (1969) 361-366

CLOSED CONFORMAL VECTOR FIELDS

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1. Introduction

Let M be a connected m-dimensional Riemannian manifold with metric g. A vector field X on M is conformal if and only if

$$L_X g = -(2/m)\delta\xi \cdot g, \tag{1.1}$$

where L_X denotes the Lie derivation with respect to X, $\xi = g(X, \cdot)$ is the covariant form of X with respect to g, and $\delta\xi$ is the corresponding codifferential form of ξ. Let d be the exterior differential operator, and Q the Ricci operator which is defined on the 1-form ξ by $Q\xi = R_1(X, \cdot)$, R_1 being the Ricci tensor.

If M is a connected 2-dimensional Riemannian manifold with constant scalar curvature $S > 0$, and M admits a conformal non-Killing vector field, then M is globally isometric with a sphere (cf. [1]).

Next let M be a connected, compact Riemannian manifold of dimension $m > 2$ with constant scalar curvature $S > 0$. Then M is globally isometric with a sphere, if M admits a conformal non-Killing vector field X and any one of the following conditions is satisfied:

(a) *M is an Einstein space* [6], [8],

(b) *the Ricci tensor is parallel* [5],

(c) *trace Q^2 is constant* [4],

(d) *$Qd\delta\xi = kd\delta\xi$ for some constant k* [9],

(e) *$Qd\delta C_0^*(M) \subset d\delta C_0^*(M)$, where $C_0^*(M)$ denotes the space of covariant forms of all conformal vector fields on M* [1],

(f) *ξ is an exact form* [4].

(a) and (e) have been proved independently. Conditions (d) and (e) are related to (f). If ξ is exact, then ξ vanishes at some point of M. The first theorem of this note is a generalization of (f).

Theorem 1. *Let M be a connected compact Riemannian manifold with the constant scalar curvature $S > 0$. Then M is globally isometric with a sphere if M admits a closed conformal vector field X which satisfies one of the conditions:*

(i) *the harmonic part h of ξ vanishes at some point of M,*

(ii) *X vanishes at some point of M,*

Communicated by K. Yano, June, 24, 1968.