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# HOMOGENEOUS EINSTEIN SPACES OF DIMENSION FOUR

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## Introduction

Given a manifold $M$ with a Riemannian metric $g_{ij}$, Riemannian curvature tensor $R_{ijkl}$, and Ricci tensor $R_{ij} = \sum_k R_{kikj}$, then $M$ is defined to be an Einstein space if $R_{ij} = \lambda g_{ij}$ for some scalar function $\lambda$ on $M$. In fact, $\lambda = R/n$, where $R = \sum_i R_{ii}$ is the scalar curvature of $M$ and $n = \dim M$. It is a classical theorem that, for $n \geq 3$, $\lambda$ is a constant if $M$ is connected. A general problem of Riemannian geometry is the determination of all Einstein spaces.

It is also of interest to consider pseudo-Riemannian metrics $g_{ij}$. Indeed, in the case of dim $M = 4$ and for a Lorentzian metric, (signature $+ + + -$ say), the equations $R_{ij} = \lambda g_{ij}$ are specializations of Einstein field equations of General Relativity: $R_{ij} - \frac{1}{2} g_{ij} = T_{ij}$, where $T_{ij}$ is some tensor field with variously specified properties.

From the physicist's point of view $M$ is often not specified. Indeed $M$ is often, on first consideration, taken to be an open subset of Euclidean space and the Einstein equations are solved locally. However, our point of view differs from theirs in that we desire to know whether a given manifold can be an Einstein space. This question has interest only for dim $M \geq 4$, since all two dimensional Riemannian manifolds are Einstein spaces, and any three-dimensional Einstein space necessarily has constant curvature.

A fairly comprehensive list of known Einstein spaces is given by M. Berger [1, pp. 41, 42] and J. A. Wolf [9].

A natural starting point for the determination of Einstein spaces is with four-dimensional homogeneous Riemannian spaces. It is the purpose of this paper to determine all such spaces which are Einstein spaces. A homogeneous space $M$ can be represented as a quotient $G/H$, where $G$ is a transitive group of isometries on $M$ and $H$ is the isotropy group at some point $p_0 \in M$. In Chapter II it will be seen that when $G$ is large enough, for example when $\dim H > 1$, then the solution of the problem is fairly easy. However, when $G$

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