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DEFORMATIONS OF SUBALGEBRAS OF LIE ALGEBRAS

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Introduction

In this paper we shall be concerned with "deformations" of subalgebras of Lie algebras. More generally, we are interested in deformations of subalgebras in which the ambient Lie algebra is also allowed to vary. Precisely, we consider the following situation. Let $\mathscr{M}$ be the algebraic set of all Lie algebra multiplications on a finite-dimensional vector space V (taken over $\boldsymbol{R}$ or $\boldsymbol{C}$ for simplicity), and $\Gamma_n(V)$ the Grassmann manifold of all n-dimensional subspaces of V. Let $\mathscr{S}$ be the algebraic subset of $\mathscr{M} \times \Gamma_n(V)$ consisting of all pairs (η, W) such that W is a subalgebra of the Lie algebra (V, η). Let $\mathfrak{g} = (V, \mu)$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. We are interested in geometric properties of $\mathscr{S}$ in a neighborhood of $(\mu, \mathfrak{h})$. Our main result, Theorem 2.5, states that if the Lie algebra cohomology space $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ vanishes, then a neighborhood of $(\mu, \mathfrak{h})$ in $\mathscr{S}$ is isomorphic (as an analytic space) to the product of a neighborhood of μ in $\mathscr{M}$ and an open ball in $\boldsymbol{R}^k$ (or $\boldsymbol{C}^k$), where $k = \dim Z^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$.

As easy consequences of Theorem 2.5, we obtain the results discussed in (a)–(c) below. These results complement and extend the results of two earlier papers [11], [12].

(a) Let $\mathfrak{h} \subset \mathfrak{g}$ be as above. Then $\mathfrak{h}$ is a *weakly stable* subalgebra of $\mathfrak{g}$ if, roughly speaking, for every one-parameter family $(\mathfrak{g}_t) = (V, \mu_t)$ of Lie algebra structures on V with $\mathfrak{g}_0 = \mathfrak{g}$, there exists a one-parameter family $(\mathfrak{h}_t)$ of subspaces of V with $\mathfrak{h}_0 = \mathfrak{h}$ such that $\mathfrak{h}_t$ is a subalgebra of $\mathfrak{g}_t$ for t sufficiently small. It follows from Theorem 2.5 that if $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$, then $\mathfrak{h}$ is weakly stable.

(b) Let $\mathfrak{h} = (U, \eta)$ and $\mathfrak{g} = (V, \mu)$ be Lie algebras, and $\mathscr{N}$ (resp. $\mathscr{M}$) the set of all Lie multiplications on U (resp. V). A homomorphism $\rho\colon \mathfrak{h} \to \mathfrak{g}$ is *stable* if, for every $\eta' \in \mathscr{N}$ near η and every $\mu' \in \mathscr{M}$ near μ, there exists a homomorphism $\rho'\colon (U, \eta') \to (V, \mu')$ which is near ρ. We show that ρ is stable if $H^2(\mathfrak{h}, \mathfrak{g}) = 0$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and ρ the inclusion map, we obtain a strengthened form of Theorem 6.2 of [11] on stable subalgebras.

(c) Let $\mathfrak{h} \subset \mathfrak{g}$ be Lie algebras. If $(\mathfrak{h}_t)$ is a one-parameter family of subalgebras of $\mathfrak{g}$ with $\mathfrak{h}_0 = \mathfrak{h}$, then it was shown in [12] that the "initial tangent vector" of the family $(\mathfrak{h}_t)$ is an element of $Z^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$. We show that if $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h}) = 0$, then every $\alpha \in Z^1(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ occurs as the initial tangent vector of a one-parameter family of subalgebras. In a sense, the elements of $H^2(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$

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