J. DIFFERENTIAL GEOMETRY
3 (1969) 257-288

SINGULAR HOMOLOGY OVER Z ON TOPOLOGICAL MANIFOLDS

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0. Introduction

The differentiable case. On an arbitrary connected, differentiable manifold M_n of class C^∞, there always exists a real-valued nondegenerate (abbreviated ND) function f of class C^∞ with the following properties:

(a) *For each value c of f the subspace*

$$f_c = \{p \in M_n \,|\, f(p) \leq c\} \tag{0.1}$$

of M_n is compact.

(b) *The function f has different values a at different critical points.*

(c) *There is just one critical point of f of index* 0.

That such a function f exists on a manifold M_n of class C^∞ is established in the compact case in [12]. For the non-compact case see Theorem 23.5 and Lemma 22.4 of [1].

Singular homology groups on subspaces of M_n are understood in the sense of Eilenberg [2]. See also Part III of [1]. In this paper these groups are taken over Z, the ring of integers. With each critical point of a ND f we shall associate "relative numerical invariants"[1] such that the following is true:

Theorem 0.1. *There exists an inductive group-theoretic mechanism by virtue of which relative numerical invariants "associated"[2] with the critical points of f on f_c determine, up to an isomorphism, the singular homology groups over Z of the subspace f_c of M_n.*

The results in this paper were abstracted in part in Appendix III of [1]. In preparation for this paper a preliminary paper [3] has been written. Paper [3] is concerned with quotients A/W of a finitely generated abelian group A by a cyclic subgroup W of A. Given the *invariants* of A, namely the torsion coefficients of the torsion subgroup $\mathscr{T}$ of A, and the dimension[3] β of a free group $\mathscr{B}$ "complementary" in A to $\mathscr{T}$, paper [3] makes explicit a simple mechanism

Received April 26, 1969. The research of the first author was supported in part by U. S. Army Research Office-Durham grant DA-31-124-ARO-D-455, and that of the second author by National Science Foundation grant GP-8640.

[1] Defined in §6.

[2] Associated as in Condition 7.1.

[3] β is termed the "Betti number" of A and $\mathscr{B}$ a "Betti subgroup" of A.