J. DIFFERENTIAL GEOMETRY
3 (1969) 237-244

ISOMETRIC IMMERSIONS IN SYMMETRIC SPACES

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1. Introduction

Let M and $\bar{M}$ be complete Riemannian manifolds (we make this assumption throughout the paper). Denote the sectional curvatures of M and $\bar{M}$ by K and $\bar{K}$, respectively. R. Hermann [4] has proved the following theorem.

Theorem A. *Suppose that $\bar{M}$ is simply connected, M is isometrically immersed as a closed submanifold of $\bar{M}$, and $K \leq \bar{K} \leq 0$. Then $H_i(M) = 0$ for $i > \dim \bar{M} - \dim M$. (Here the homology groups have coefficients in an arbitrary field.)*

It follows from Theorem A that M cannot be immersed in $\bar{M}$ if $\dim \bar{M} < 2 \dim M$ and M is compact. In fact E. Stiel [10] has proved the following result.

Theorem B. *Suppose M is compact and $\bar{M}$ is simply connected. If $\bar{K}$ is constant, $\bar{K} \leq 0$, $K \leq 0$ and $\dim \bar{M} < 2 \dim M$, then M cannot be isometrically immersed in $\bar{M}$.*

In this paper we show that the hypotheses of Theorem A can be weakened provided $\bar{M}$ is a symmetric space. At the same time we obtain a generalization of Stiel's theorem. An example of our results is the following.

Theorem (3.4). *Suppose M is isometrically immersed as a closed submanifold of $\bar{M}$, $\bar{M}$ is a simply connected symmetric space with nonpositive curvature, and $\sup K \leq \min \bar{K} - \max \bar{K}$. Then M has the homotopy type of a CW-complex with no cells of dimension greater than $\dim \bar{M} - \dim M$.*

We also give variations of the above theorem under the hypotheses that M is a minimal variety of $\bar{M}$ or that $\bar{M}$ is not simply connected. Furthermore we consider the case when $\bar{M}$ has positive curvature, and generalize some results of Ôtsuki [9].

2. The Hessian of the distance function

In this section we assume that M is isometrically immersed as a closed C^∞ submanifold of $\bar{M}$. Let M_p and $\bar{M}_p$ be the tangent spaces of M and $\bar{M}$ at a point $p \in M$, and write $\bar{M}_p = M_p \oplus M_p^\perp$. We denote by $\langle , \rangle$ the metric tensor of M or $\bar{M}$, and by $R_{xy}(x, y \in M_p)$ and $\bar{R}_{zw}(z, w \in \bar{M}_p)$ the curvature operators of

Received July 17, 1968. This work was partially supported by National Science Foundation grant GP-6974.