J. DIFFERENTIAL GEOMETRY
3 (1969) 213-229

MORSE THEORY OF CERTAIN SYMMETRIC SPACES

S. RAMANUJAM

Introduction

T. T. Frankel applied Morse theory to the classical groups, which we shall denote by G, and Stiefel manifolds [1] by taking the matrix representation of the classical groups and using the "trace function" as Morse function. The present author [4] obtained a Morse decomposition for certain symmetric spaces G/K by using methods similar to those of Frankel with the same trace function since those symmetric spaces are imbedded in the group G. M. Takeuchi [6] has considered the same problem and others from a more general point of view.

The purpose of this paper is to eliminate the heavy manipulation of [4] and to point out alternate methods with the symmetric spaces described in terms of matrices, so that this paper differs from [4] in the following three ways:

First, the critical submanifolds of G/K are shown to be the intersection of the space G/K and the critical submanifolds of G.

Secondly, the indices of the critical submanifolds of G/K are immediately obtained from that of the critical submanifolds of G.

Thirdly, "Floyd Theorem B" is used not to a great extent but only for the case $U(2n)/Sp(n)$.

1. Preliminaries

We briefly describe how the coset space G/K arises as a symmetric space. Let G be a compact connected Lie group with a left and right invariant Riemannian metric. (In all our discussions G will denote the classical groups, i.e., $SO(n)$, $U(n)$ or $Sp(n)$.) Let θ be an involution on G, K the full fixed set, and K' the identity component of K. Then G/K is a symmetric space.

Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}$ decomposes into a natural direct sum, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \{x \in \mathfrak{g} \mid s(x) = x\}$ and $\mathfrak{p} = \{x \in \mathfrak{g} \mid s(x) = -x\}$ i.e., into eigenspaces of eigenvalue $+1$ and -1 for s, the differential action induced by θ on $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{p}$ be a maximal subalgebra of $\mathfrak{p}$. Then $\mathfrak{h}$ is abelian and is called *Cartan subalgebra*.

Define $\eta\colon G \to G$ by $\eta(g) = g\cdot\theta(g^{-1})$. Then $\eta(gK) = \eta(g)$, which means η is constant along the left cosets of K. Hence, η induces a map $\eta_*\colon G/K \to G$.

Received April 11, 1968. This research was supported by NSF Grant GP-5874.