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CONVEXITY OF PARTIAL DIFFERENTIAL OPERATORS

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In [3, Chapter VIII], Hörmander has proved some inequalities for some partial differential operators by assuming certain convexity conditions. We shall obtain some generalization of these inequalities under more general convexity conditions. However, our primary aim has been to insert some geometric meaning into the formulas of Chapter VIII of [3] via the concept of the characteristic vector field of a function on a Grassmann bundle. This geometric interpretation suggests our generalization whose proof then goes via the analytic techniques of Hörmander. The core of the proof is the same as in [3], but the modifications needed to treat the more general case are non-trivial.

Let Q be a bounded open subset of R^p. Hörmander's convexity condition, for a linear partial differential operator defined on $\bar{Q}$, involves a real valued function φ on $\bar{Q}$. We replace φ by a map $\emptyset$ which assigns to each x in $\bar{Q}$ a nonsingular linear transformation of R_x^{p+1} into R_x^{p+2} (where R_x^q is the tangent space of R^q at $x \in R^q$). In the case considered in [3], our $\emptyset$ is obtained from the φ of [3] by

$$\emptyset(x)\frac{\partial}{\partial u_i}(x) = \frac{\partial}{\partial u_i}(x) + \frac{\partial \varphi}{\partial u_i}(x)\frac{\partial}{\partial u_{p+2}}(x) \quad (1 \le i \le p) ,$$
$$\emptyset(x)\frac{\partial}{\partial u_{p+1}}(x) = \frac{\partial}{\partial u_{p+1}}(x) .$$

We explain here why a $\emptyset$ of our type gives an appropriate generalization of the φ which occurs in [3]. Let $P(D)$ be an m-th order linear partial differential operator defined on $\bar{Q}$, and P_m the symbol of $P(D)$. P_m is usually considered as a function on $Q \times C^p$, but instead, we shall consider it as a function on a subset of $G_p(R^{p+2})$, where $G_p(R^q)$ denotes the Grassmann bundle of all p-planes at all points of R^q. Then $\emptyset$ induces, in an obvious way, a map $\emptyset'$ of $G_p(R^{p+1}) \to G_p(R^{p+2})$. We consider the function $P_m \circ \emptyset'$, defined on a subset of $G_p(R^{p+1})$, which has a characteristic vector field V. We assert that the essential function which enters into the convexity conditions of Chapter VIII of [3] is the function H, defined on a subset of $G_p(R^{p+1})$, by

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