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SYMMETRIES OF SURFACES OF CONSTANT WIDTH

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1. Introduction

A *hypersurface of constant width* in Euclidean space E^n is a compact continuous $(n-1)$-dimensional submanifold enclosing an open convex set and having the property that the distance between the two supporting hyperplanes having a given unit vector ξ as normals is independent of ξ.

Examples of such hypersurface in E^2 are the Reuleux triangle and a parallel curve of constant distance from a Reuleux triangle [4, p. 199]; these curves are continuous and of class C^1 respectively.

Assume that the coordinate origin of E^n is inside the region enclosed by the hypersurface. Given a unit vector ξ, let $h(\xi)$ denote the distance from the origin to the supporting hyperplane having ξ as normal and such that the vector from the origin to the point of contact makes an acute angle with ξ. $h(\xi)$ is the Minkowski support function and

$$h(\xi) + h(-\xi) = \text{constant} = 2a$$

is necessary and sufficient for the hypersurface to be of constant width.

In E^2 we may take $\xi = (\cos\theta, \sin\theta)$ and $h = a + b\cos 3\theta$ $(0 < 8b < a)$. The resulting curve

$$x_1 = h\cos\theta - \frac{dh}{d\theta}\sin\theta\ , \qquad x_2 = h\sin\theta + \frac{dh}{d\theta}\cos\theta$$

is analytic and of constant width—an analytic version of the Reuleux triangle. If we rotate this curve in E^n about an $(n-2)$-dimensional axis perpendicular to the line $\theta = 0$, we obtain an analytic surface of constant width in E^n, which is not a sphere.

The hypersurfaces just described admit many symmetries—orthogonal transformations (including the improper ones) which, when combined with a translation, carry the hypersurface into itself. In this paper we show that there exist in E^n analytic hypersurfaces of constant width, which do not admit any symmetries other than the identity.

In the final section we establish which closed subgroups of the proper and

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