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NON-HYPERELLIPTIC RIEMANN SURFACES

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1. Let W be a Riemann surface of genus $g \geq 2$. Abel's Theorem gives an analytic embedding, with respect to an arbitrary base point, of W as a submanifold of its Jacobi variety $J(W)$ (see Gunning [1, p. 161]). Denote by W_r the linear equivalence classes in $J(W)$ of divisors consisting of r points in W. Then $W = W_1$ and W_r is a subvariety of dimension r for $0 \leq r \leq g$. More generally, let W_r^a denote the set of all linear equivalence classes of divisors of degree r in $J(W)$, which admit at least a linearly independent meromorphic function, or equivalently, the set of all line bundles on W of Chern class r which admit at least a linearly independent analytic section. Then $W_r = W_r^1$, and W_r^a is a subvariety of $J(W)$. Alan Mayer [4] showed that $\dim W_r^a \leq r - 2a + 2$ (provided $1 \leq r \leq g - 1$, $a \geq 2$, and $r - 2a + 2 \geq -1$) and that the maximum is in fact attained whenever W is hyperelliptic. He then conjectured that the converse was true, and this is our main result.

Theorem 1. *If W is not hyperelliptic, then* $\dim W_r^a \leq r - 2a + 1$ (*provided* $1 \leq r \leq g - 1$, $a \geq 2$, *and* $r - 2a + 1 \geq -1$).

Thus surfaces which are not hyperelliptic have fewer "special" divisors than those which are.

2. Before proceeding to the proof of Theorem 1, it may be of interest to see how two classical theorems on non-hyperelliptic surfaces may be deduced from this result.

Clifford's Theorem. *If W is not hyperelliptic, then no translate of W_r is contained in $-W_r$, for $1 \leq r \leq g - 2$.*

Proof. The set of all elements x such that the translate of $-W_r$ by x is contained in W_r is the set $W_r \ominus -W_r = W_{2r}^{r+1}$ (see § 3). Hence the theorem is equivalent to the assertion that if W is not hyperelliptic then W_{2r}^{r+1} is empty. If $2r \leq g - 1$, Theorem 1 states that $\dim W_{2r}^{r+1} \leq -1$. If $2r \geq g - 1$, let k be the divisor class in $J(W)$ of an abelian differential. Then $k - W_{2r}^{r+1} = W_{2g-2-2r}^{g-r}$ by the Riemann-Roch Theorem, and since $2 \leq 2g - 2 - 2r \leq g - 1$ this is the same as the first case.

Nöther's Theorem. *If W is not hyperelliptic, then every quadratic differential on W can be written as the sum of* (*three*) *products of abelian differentials.*

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