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SYMMETRIC SPACES WHICH ARE REAL COHOMOLOGY SPHERES

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This is a survey in which we collate some known results using semi-standard techniques, dropping the condition of simple connectivity in Kostant's work [2] and proving

Theorem 1. *Let M be a compact connected riemannian symmetric space. Then M is a real cohomology* ($\dim M$)*-sphere if and only if*

(1) *M is an odd dimensional sphere or real projective space; or*

(2) $M = \bar{M}/\Gamma$ *where* (a) $\bar{M} = \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_m}$, $r_i > 0$, *product of $m \geq 1$ even dimensional spheres, and* (b) *Γ consists of all $\gamma = \gamma_1 \times \cdots \times \gamma_m$, where γ_i is the identity map or the antipodal map of $\mathbf{S}^{2r_i}$, and the number of γ_i which are antipodal maps, is even; or*

(3) $M = \mathbf{SU}(3)/\mathbf{SO}(3)$ *or* $M = \{\mathbf{SU}(3)/\mathbf{Z}_3\}/\mathbf{SO}(3)$; *or*

(4) $M = \mathbf{O}(5)/\mathbf{O}(2) \times \mathbf{O}(3)$, *non-oriented real grassmannian of 2-planes through* 0 *in* $\mathbf{R}^5$.

In (2) we note $\pi_1(M) = \Gamma \cong (\mathbf{Z}_2)^{m-1}$; in particular the even dimensional spheres are the case $m = 1$. In (3) we note that the first case is the universal 3-fold covering of the second case. In (4) we have $\pi_1(M) \cong \mathbf{Z}_2$.

Theorem 1 is based on a series of lemmas which can be pushed, with appropriate modification, to the case of a real cohomology n-sphere of dimension greater than n. *Here we make the convention that a 0-sphere is a single point.* By using a cohomology theory which satisfies the homotopy axiom (such as singular theory) we can also drop the requirement of compactness. Thus we push the method of proof of Theorem 1 and obtain

Theorem 2. *Let M be a connected riemannian symmetric space. Then M is a real cohomology n-sphere, $0 \leq n \leq \dim M$, if and only if $M = M' \times M''$ where (α) M'' is a product whose $l \geq 0$ factors are euclidean spaces and irreducible symmetric spaces of noncompact type, and (β) M' is one of the following spaces.*

(1) $M' = \bar{M}/\Gamma^\theta$, *where $\bar{M} = \mathbf{S}^{2r_1} \times \cdots \times \mathbf{S}^{2r_m}$ is the product of $m \geq 0$ spheres of positive even dimensions $2r_i$, $\Gamma \cong (\mathbf{Z}_2)^m$ consists of all $\gamma_1 \times \cdots \times \gamma_m$ such that γ_i is the identity or antipodal map on $\mathbf{S}^{2r_i}$, θ is any one of the 2^m characters on Γ, and Γ^θ is the kernel of θ. Express $\theta = \theta_{i_1} \cdots \theta_{i_s}$, where*

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