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A FIBRE BUNDLE DESCRIPTION OF TEICHMÜLLER THEORY

CLIFFORD J. EARLE & JAMES EELLS

1. Introduction

(A) In this paper we prove the theorems which we announced in [14] concerning the diffeomorphism groups of a closed surface, and, in addition, the corresponding theorems for the diffeomorphism groups of the closed non-orientable surfaces. Our method is to construct a certain principal fibre bundle, whose total space is the space of smooth conformal structures of a closed surface, whose base is a Teichmüller space, and whose structural group is a subgroup of the diffeomorphism group of the surface. Our bundle has the further property that its tangent bundle sequence embodies the infinitesimal deformation of structure theory (for surfaces) of Kodaira-Spencer [22].

Set theoretically, the construction of our bundle is a modification of the Ahlfors-Bers development of Teichmüller theory. To show that we have produced a topological fibre bundle, we need a new theorem about the continuity of solutions to Beltrami equations with smooth coefficients (see § 3). We have provided a fairly detailed account of our construction, because even where it closely follows the Ahlfors-Bers developments, certain adjustments are needed. Consequently we believe that the reader will find the paper relatively self-contained. For expositions of Teichmüller theory, and for guides to the literature, we refer to Ahlfors [2], Bers [6], Rauch [26], and Teichmüller [30].

(B) We now formulate precisely our main results. Let X be an oriented smooth (= class C^∞) 2-dimensional manifold which is compact and without boundary. We denote by $\mathbf{D}(X)$ the topological group of all orientation preserving diffeomorphisms of X, endowed with the C^∞-topology of uniform convergence of differentials of all orders; $\mathbf{D}_0(X)$ is the subgroup consisting of the diffeomorphisms which are homotopic to the identity. (We shall find later that $\mathbf{D}_0(X)$ is the arc component in $\mathbf{D}(X)$ of the neutral element.)

We denote by $\mathbf{M}(X)$ the space of smooth complex structures on X compatible with its given orientation, and give $\mathbf{M}(X)$ the C^∞-topology. Then (viewing the elements of $\mathbf{M}(X)$ as smooth tensor fields on X) we have a natural action

$$\mathbf{M}(X) \times \mathbf{D}(X) \to \mathbf{M}(X) .$$

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