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EQUIVARIANT K-THEORY AND COMPLETION

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1. Introduction

It was shown in [3] that, for any *finite* group G, the completed character ring $R(G)^\wedge$ was isomorphic to $K^*(B_G)$ where B_G denotes a classifying space for G. The corresponding result for compact *connected* Lie groups was established in [2], and a combination of the methods of [2] and [3] (together with certain basic properties of $R(G)$ given in [18]) can be used to derive the theorem for general compact Lie groups. Such a proof however would be extremely lengthy, the worst part being in fact the treatment for finite groups where one climbs up via cyclic and Sylow subgroups.

The purpose of this paper is to give a new and much simpler proof of the theorem about $K^*(B_G)$ which applies directly to all compact Lie groups G. The main feature of our new proof is that we generalize the whole problem in a rather natural way by working with the equivariant K-theory developed in [17]. We shall formulate and prove a general theorem about the completion $K_G^*(X)^\wedge$ for any compact G-space X. The theorem about $R(G)$ then follows by taking X to be a point.

The proof consists of four steps. First we deal with the case when $G = \boldsymbol{T}$ is the circle group. Because of the simple model for B_T given by the (infinite) complex projective space this case is easily dealt with directly. The second step is to pass from the circle to a general torus, and this is done in an obvious way by induction on the dimension of the torus. The third and key step shows how to reduce the case of the unitary group $U(n)$ to its maximal torus; this depends on the analytical methods, using elliptic operators, developed in [6]. The fourth and final step reduces the case of a general group G to the case of a unitary group by means of an embedding $G \subset U$; we replace the G-space X by the U-space $Y = U \times_G X$. Thus, *even if we are only interested in the case when G is finite and X is a point*, we are forced at this stage to consider the Lie group U and the U-space U/G.

Using the spectral sequence of [17] it would in principle be possible to pass from the case of a point to general X. However, as we have just explained, there is nothing to be gained by this procedure because the proof we give applies naturally to the general case.

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