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GROWTH OF FINITELY GENERATED SOLVABLE GROUPS

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This note is intended as an addendum to the preceeding paper [1] by J. A. Wolf. We will prove the following

Theorem. *Let Γ be a solvable group which is not polycyclic, and S a finite set of generators for Γ. Then there exists an exponential lower bound*

$$g_S(m) \geq (\text{constant})^m > 1$$

for the growth function g_S of Γ.

Briefly, Γ has "exponential growth." For definitions and explanations the reader is referred to [1]. Note that the results of [1] provide a partial answer to a problem which was posed by the author in Amer. Math. Monthly **75** (1968) 685-686.

The proof will be based on the study of a group extension

$$1 \longrightarrow A \longrightarrow B \xrightarrow{\varphi} C \longrightarrow 1 ,$$

where we will always assume that A is abelian and that B is finitely generated. Let Z denote the ring of integers.

Lemma 1. *If B does not have exponential growth, then for each $\alpha \in A$ and $\beta \in B$ the set of all conjugates $\beta^k \alpha \beta^{-k}$, with $k \in Z$, spans a finitely generated subgroup of A.*

Proof. For each sequence $i_1, i_2, \cdots, i_m$ of 0's and 1's consider the expression

$$\beta\alpha^{i_1}\beta\alpha^{i_2} \cdots \beta\alpha^{i_m} \in B .$$

If these 2^m expressions all represented distinct elements of B, then the growth function g_S of B, computed using any set S of generators for B which contains both β and $\beta\alpha$, would satisfy

$$g_S(m) \geq 2^m .$$

But this would contradict the hypothesis. Hence there must exist a nontrivial relation of the form

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