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GROWTH OF FINITELY GENERATED SOLVABLE GROUPS AND CURVATURE OF RIEMANNIAN MANIFOLDS

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1. Introduction and summary

If a group Γ is generated by a finite subset S, then one has the "growth function" g_S, where $g_S(m)$ is the number of distinct elements of Γ expressible as words of length $\leq m$ on S. Roughly speaking, J. Milnor [9] shows that the asymptotic behaviour of g_S does not depend on choice of finite generating set $S \subset \Gamma$, and that lower (resp. upper) bounds on the curvature of a riemannian manifold M result in upper (resp. lower) bounds on the growth function of $\pi_1(M)$. The types of bounds on the growth function are

$$\textit{polynomial growth of degree} \leq E: \quad g_S(m) \leq c \cdot m^E ,$$
$$\textit{exponential growth:} \quad u \cdot v^m \leq g_S(m) ,$$

where c, u and v are positive constants depending only on S, $v > 1$, and m ranges over the positive integers.

In § 3 we show that, *if a group Γ has a finitely generated nilpotent subgroup Δ of finite index, then it is of polynomial growth, and in fact $c_1 m^{E_1(\Delta)} \leq g_S(m) \leq c_2 m^{E_2(\Delta)}$, where $0 < c_1 \leq c_2$ are constants depending on the finite generating set $S \subset \Gamma$, and $E_1(\Delta) \leq E_2(\Delta)$ are positive integers specified in* (3.3) *by the lower central series of Δ*. In § 4 we consider a class of solvable groups which we call "polycyclic"; Proposition 4.1 gives eleven characterizations, all useful in various contexts; finitely generated nilpotent groups are polycyclic. We prove that *a polycyclic group, either has a finitely generated nilpotent subgroup of finite index and thus is of polynomial growth, or has no such subgroup and is of exponential growth.* We also give a workable criterion for deciding between the two cases. Applying a result of Milnor [10] which says that a finitely generated nonpolycyclic solvable group is of exponential growth, we conclude that *a finitely generated solvable group, either is polycyclic and has a nilpotent subgroup of finite index and is thus of polynomial growth, or has no nilpotent subgroup of finite index and is of exponential growth.*

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