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COMPACT REAL FORMS OF A COMPLEX SEMI-SIMPLE LIE ALGEBRA

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Introduction

This paper gives a new proof of an old theorem, the existence of a compact real form of a complex semi-simple Lie algebra. The theorem is a consequence of the classification of real simple Lie algebras by E. Cartan in 1914 [1]. Later H. Weyl [8] gave an intrinsic proof based on the detailed structure theory of semi-simple Lie algebras. Our proof, which is based on a suggestion of Cartan [2, p. 23], is geometric in nature. The *only* results from the theory of Lie algebras which we have used are the facts that if $\mathfrak{g}$ is a semi-simple Lie algebra, then the center of $\mathfrak{g}$ is $\{0\}$ and every derivation of $\mathfrak{g}$ is inner. On the debit side, however, our proof uses an elementary lemma from algebraic geometry and does involve one long and unedifying computation.

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1. Preliminaries

$\boldsymbol{R}$ (resp. $\boldsymbol{C}$) denotes the field of real (resp. complex) numbers. If S is a set, then S^m denotes the m-fold Cartesian product $S \times \cdots \times S$. N_n denotes the set $\{1, \cdots, n\}$. If W is a vector space over $\boldsymbol{C}$, then $W^{\boldsymbol{R}}$ is the real vector space obtained from W by restriction of scalars. If V is a vector space over $\boldsymbol{C}$, then $A^m(V, V)$ denotes the vector space of all alternating m-linear maps of V^m into V.

Let $B = \{e_1, \cdots, e_n\}$ be a basis of V. If $\varphi \varepsilon A^m(V, V)$, we write $\varphi(e_{a_1}, \cdots, e_{a_m}) = \sum_{j=1}^{m}(\varphi_{a_1 \cdots a_m j})e_j$. The $\varphi_{a_1 \cdots a_{m+1}}$ are the "coordinates" of φ with respect to the basis B, and we often write $\varphi = \varphi(_{a_1 \cdots a_{m+1}})$. The basis B determines a positive definite Hermitian inner product on $A^m(V, V)$ as follows: If φ, $\psi \varepsilon A^m(V, V)$, then $\langle \varphi, \psi \rangle = \sum_{\boldsymbol{a}} \varphi_{\boldsymbol{a}} \bar{\psi}_{\boldsymbol{a}}$, where the sum is taken over all $\boldsymbol{a} = (a_1, \cdots, a_{m+1}) \varepsilon (N_n)^{m+1}$ and the bar denotes complex conjugation. Let $\langle \varphi, \psi \rangle_r$ denote the real part of the complex number $\langle \varphi, \psi \rangle$. Then $(\varphi, \psi) \to \langle \varphi, \psi \rangle_r$ is a positive definite inner product on the real vector space $A^m(V, V)^{\boldsymbol{R}}$. For $\varphi \varepsilon A^m(V, V)$ we write $\|\varphi\|^2 = \langle \varphi, \varphi \rangle = \langle \varphi, \varphi \rangle_r$.

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