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FLAT MANIFOLDS WITH PARALLEL TORSION

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1. Consider a linear connection on a smooth manifold. The connection is *flat*, if the curvature tensor R is zero. If the torsion tensor T has vanishing covariant derivative, the *torsion* is said to be *parallel*. A linear connection is *complete*, if every geodesic can be defined for any real value of the affine parameter. In this note the following structure theorem for smooth manifolds admitting a complete flat connection with parallel torsion is proved: *Any such manifold is the orbit space of a simply connected Lie group G under a properly discontinuous and fixed-point free action of a subgroup of the affine group of G.* This Theorem includes the classical cases of flat Riemannian manifolds and flat affine manifolds (Auslander and Markus [2]), where the torsion is assumed to be zero and G turns out to be $\boldsymbol{R}^n$, and also generalizes a theorem of Hicks [9, Theorem 6] for complete connections with trivial holonomy group and parallel torsion tensor, stating that a manifold with such a connection is homogeneous. We consider the case where the curvature vanishes, without requiring the holonomy group to be trivial. In the last section we study the homotopy group of flat manifolds with parallel torsion and give characterizations for such manifolds to be Eilenberg-MacLane spaces of type $K(\pi, 1)$.

2. Let G be a Lie group and Aut G the group of continuous automorphisms of G. The affine group of G [1] is the semi-direct product $A(G) = G \cdot \operatorname{Aut} G$ with the multiplication $(g_1, \alpha_1) \cdot (g_2, \alpha_2) = (g_1\alpha_1(g_2), \alpha_1\alpha_2)$ for $g_1, g_2 \in G$ and $\alpha_1, \alpha_2 \in \operatorname{Aut} G$, It has a Lie group structure, and acts on G by $(g, \alpha)\cdot x = g\alpha(x)$ for $(g, \alpha) \in A(G)$, $x \in G$. In the case of the additive vector group $\boldsymbol{R}^n$ this is the affine group $A(n) = \boldsymbol{R}^n \cdot GL(n, \boldsymbol{R})$ with its standard action on $\boldsymbol{R}^n$.

Let G be connected and consider the linear connection on G defined by the left invariant vector fields [8], [9]. The parallel transport is the effect of the left translations on the tangent vectors of G, and hence clearly independent of paths; the connection is thus flat. The geodesics through the identity element $e \in G$ are the 1-parameter subgroups of G and thus defined for any real value of the affine parameter. All geodesics are translates of geodesics through e and thus the connection is complete. The parallelity of the torsion is easily checked [7], [9].

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