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A MORSE FUNCTION ON GRASSMANN MANIFOLDS

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Studying the critical sections of a convex body Wen Tsun Wu has obtained in [2] a Morse function on a Grassmann manifold. In the sequel it will be shown that another function may be obtained by composing the embedding of this manifold into a projective space with the well known Morse function of the projective space; our work is valid only for the real and complex fields.

1. The homology of the Grassmann manifold $G_{p,q}$ of all the p-planes of codimension q which pass through a fixed point 0 in an affine space A^n of dimension $n = p + q$ was determined in 1934 by Ch. Ehresmann who gave a cell subdivision of $G_{p,q}$. The number of cells in his subdivision is the number $N = \binom{n}{p}$ of combinations of p elements of the set $\{1, \cdots n\}$; such a combination $\sigma = (\sigma_1, \cdots, \sigma_p)$ where $1 \leq \sigma_1 < \cdots < \sigma_p \leq n$ is called a Schubert symbol. In the cell-subdivision of $G_{p,q}$, with each symbol σ one associates a cell of dimension

$$d(\sigma) = (\sigma_1 - 1) + \cdots + (\sigma_p - p) .$$

Let us consider the lexicographical order in the set $S(p, q)$ of all the Schubert symbols which correspond to the integers p and q; this means that $\sigma = (\sigma_1, \cdots, \sigma_p) < \sigma' = (\sigma'_1, \cdots, \sigma'_p)$ if and only if for the least integer $i \leq p$ for which $\sigma_i \neq \sigma'_i$ the inequality $\sigma_i < \sigma'_i$ holds. We say that two symbols $\sigma = (\sigma_1, \cdots, \sigma_p)$ and $\sigma' = (\sigma'_1, \cdots, \sigma'_p)$ are *neighboring* if the sets $\{\sigma_1, \cdots, \sigma_p\}$ and $\{\sigma'_1, \cdots, \sigma'_p\}$ have exactly $p - 1$ elements in common, or equivalently, if they differ only in what a single element is concerned. With these conventions we observe that the number $d(\sigma)$ equals the number of those Schubert-symbols which are less than and neighboring to σ. Indeed, in order to obtain a new symbol less than and neighboring to σ, the change of σ_i in σ may be made in $\sigma_i - i$ ways by replacing σ_i with a positive integer less than σ_i and different from $\sigma_1, \cdots, \sigma_{i-1}$.

2. In the projective space P^{N-1} of dimension $N - 1$ we consider homogeneous coordinates y_σ having as indices Schubert symbols $\sigma \in S(p, q)$ instead of positive integers running from 1 to N.

It is known, for example from [1], that the function

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