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HOLOMORPHIC MAPPINGS OF COMPLEX MANIFOLDS

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1. Introduction

Schwarz's lemma, as formulated by Pick, can be stated as follows: Every holomorphic map f of the open unit disk D into itself is distance-decreasing with respect to the Poincáre-Bergman metric ds^2, i.e. $f^*(ds^2) \leq ds^2$, where the equality holding at one point of D, implies that f is an isometry. Bochner and Martin proved in their book [2] the following generalization of Schwarz's lemma to higher dimensions. Let D_n be the n-dimensional open unit ball. If f is a holomorphic map of D_m into D_n such that $f(0) = 0$, then $f(z) \leq z$ for all $z \in D_m$. In other words, every holomorphic map of D_m into D_n is distance-decreasing with respect to the Bergman metric $ds^2_{D_m}$ and $ds^2_{D_n}$ of D_m and D_n respectively. Koranyi proved [9] that if M is a hermitian symmetric space of non-compact type with the Bergman metric ds^2, and f is a holomorphic map of M into itself, then $f^*(ds^2) \leq kds^2$, where k denotes the rank of M. This is another generalized Schwarz's lemma. Ahlfors was the first to generalize Schwarz's lemma by essentially considering the curvature; his result can be stated as the following: Let M be a Riemann surface with hermitian metric ds^2_M whose Gaussian curvature is bounded above by a negative constant $-B$, and D the unit disk in C with an invariant metric ds^2_D whose Gaussian curvature is a negative constant $-A$, then every holomorphic map $f: D \to M$ satisfies $f^*(ds^2) \leq \dfrac{A}{B} ds^2_D$. Kobayashi generalized this result to higher dimensional case in his recent paper [6].

Recently Chern [5] has shown that a holomorphic map f of D_n into a n-dimensional hermitian Einstein manifold N with scalar curvature less than or equal to $-2n(n + 1)$ is volume-decreasing. Kobayashi [8] generalized the result of Chern to the case of a holomorphic mapping f from a more generalized domain M (of dimension n) into a more general image manifold N (of dimension n). This paper is devoted to a generalization of Schwarz's lemma as well as one of Chern's results in his paper [5] concerning a Laplacian formula for the ratio function of top volume elements between hermitian manifolds.

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