J. DIFFERENTIAL GEOMETRY
2 (1968) 225-232

SIMPLE CLOSED GEODESICS ON PINCHED SPHERES

WILHELM KLINGENBERG

Let M be a compact simply connected n-dimensional riemannian manifold. If the values of the sectional curvature K of M satisfy the condition $\min K : \max K > 1/4$ then M is homeomorphic to the n-sphere S^n. We therefore call such a manifold a pinched sphere. Cf. [1], [3] for the proof of this so-called sphere theorem, and [2] for a complete exposition.

By multiplying the riemannian metric of such a manifold M with an appropriate positive constant we obtain a manifold for which the relation

$$(*) \qquad 1/4 < \kappa \leq K \leq 1$$

holds. That is to say, such a manifold is curved stronger than the sphere $S^n(4)$ of constant curvature $1/4$ and curved less strongly than the unit sphere $S^n = S^n(1)$. This raises the problem of existence, on such a manifold, of simple closed geodesics of length between 2π and 4π, the length of the great circles on $S^n(1)$ and $S^n(4)$ respectively. The example of the n-dimensional ellipsoid with pairwise different axes suggests that there may be, but not more than, $n(n+1)/2$ such closed geodesics on a manifold M satisfying $(*)$.

In the present paper we shall first prove the following

Theorem 1. *On a compact simply connected n-dimensional riemannian manifold M satisfying $(*)$ there exist at least n simple closed geodesics with length in* $[2\pi, 2\pi/\sqrt{\kappa}]$.

To formulate our next result, for each integer $n \geq 2$ we define the integer $g(n)$ as $2n - s(n) - 1$ with $s(n) = n - 2^h$, $0 \leq s(n) < 2^h$. Note that for $n = 2^{2h} - 1$ we have $s(n) = (n-1)/2$ and hence $g(n) \geq (3n-1)/2$.

Theorem 2. *Let M be a compact simply connected n-dimensional riemannian manifold satisfying $(*)$ with $\kappa(\sim 0.64)$ as the solution of* $2\kappa \sin \frac{\pi}{2\sqrt{\kappa}} = 1$. *Then on M there exist $g(n)$ simple closed geodesics of length in* $[2\pi, 4\pi[$. *If $\kappa(\sim 0.46)$ is the solution of* $2\sqrt{\kappa} \sin \frac{\pi}{2\sqrt{\kappa}} = 1$, *then there exist at least $g(n-1)$ such geodesics on M.*

Suppose that on M the closed geodesics of length $< 4\pi$ are not isolated or are non-degenerate. If $(*)$ holds with $\kappa \sim 0.64$, then on M there exist exactly

Communicated April 15, 1968. This paper was completed with a support from NSF grant GP-6974.