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THE GAUSS MAP OF IMMERSIONS OF RIEMANNIAN MANIFOLDS IN SPACES OF CONSTANT CURVATURE

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Dedicated to Professor H. Hombu on his 60th birthday

0. Introduction

With an immersion x of a Riemannian n-manifold M into a Euclidean N-space E^N there is associated the Gauss map, which assigns to a point p of M the n-plane through the origin of E^N and parallel to the tangent plane of $x(M)$ at $x(p)$, and is a map of M into the Grassmann manifold $G_{n,N} = O(N)/O(n) \times O(N-n)$.

An isometric immersion of M into a Euclidean N-sphere S^N can be viewed as one into a Euclidean $(N+1)$-space E^{N+1}, and therefore the Gauss map associated with such an immersion can be determined in the ordinary sense. However, for the Gauss map to reflect the properties of the immersion into a sphere, instead of into the Euclidean space, it seems desirable to modify the definition of the Gauss map appropriately. To this end we consider the set Q of all the great n-spheres in S^N, which is naturally identified with the Grassmann manifold of $(n+1)$-planes through the center of S^N in E^{N+1}, since such $(n+1)$-planes determine unique great n-spheres and conversely.

In this paper by the Gauss map of an immersion x into S^N is meant a map of M into the Grassmann manifold $G_{n+1,N+1}$ which assigns to each point p of M the great n-sphere tangent to $x(M)$ at $x(p)$, or the $(n+1)$-plane spanned by the tangent space of $x(M)$ at $x(p)$ and the normal to S^N at $x(p)$ in E^{N+1}.

More generally, with an immersion x of M into a simply-connected complete N-space V of constant curvature there is associated a map which assigns to each point p of M the totally geodesic n-subspace tangent to $x(M)$ at $x(p)$. Such a map is called the (generalized) Gauss map. Thus the Gauss map in our generalized sense is a map: $M \to Q$, where Q stands for the space of all the totally geodesic n-subspaces in V.

The purpose of the present paper will be first to obtain a relationship among the Ricci form of the immersed manifold and the second and third fundamental forms of the immersion, and then to give a geometric interpretation of the

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