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MINIMAL IMBEDDINGS OF *R*-SPACES

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1. Introduction

Let G be a connected real semi-simple Lie group without center and U a parabolic subgroup of G. The quotient space G/U is called an *R-space*. A maximal compact subgroup K of G is transitive on G/U so that an R-space is necessarily compact. Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ be a Cartan decomposition of the Lie algebra $\mathfrak{G}$ of G with respect to the Lie algebra $\mathfrak{K}$ of K. The main purpose of this paper is to construct a natural imbedding φ of an R-space G/U into $\mathfrak{P}$ with the following properties:

(1) φ is K-equivariant;

(2) φ has minimum total curvature;

(3) If G is simple and $K/K \cap U$ is a symmetric space, then φ is isometric and $\varphi(G/U)$ is a minimal submanifold of a hypersphere in $\mathfrak{P}$ in the sense that its mean curvature normal is zero.

In general, an n-dimensional submanifold M of the hypersphere $S^N(r)$ of radius r about the origin in the Euclidean space $\boldsymbol{R}^{N+1}$ is a minimal submanifold if and only if

$$\Delta y^i = -\frac{n}{r^2} y^i \quad \text{on } M \text{ for } i = 1, \cdots, N+1 ,$$

where $(y^1, \cdots, y^{N+1})$ is a coordinate system for $\boldsymbol{R}^{N+1}$ and Δ is the Laplacian of M. For many symmetric R-spaces we verify that the Laplacian Δ for functions has no eigen-value between 0 and $-n/r^2$. We do not know whether this is true or not in general for all symmetric R-spaces.

Previously, it was known that φ has minimum total curvature if G/U is a Käehlerian C-space (Kobayashi [6]) or if G/U is a symmetric space of rank 1 (Tai [15]). For a symmetric R-space G/U, the imbedding φ has been considered by Nagano [13], and has also been conjectured to have minimum total curvature (Kobayashi [7]). The class of symmetric R-spaces includes

(i) all hermitian symmetric spaces of compact type;

(ii) Grassmann manifolds $O(p+q)/O(p) \times O(q)$, $Sp(p+q)/Sp(p) \times Sp(q)$;

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