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A UNIQUENESS THEOREM FOR MINIMAL SUBMANIFOLDS

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1. Introduction

The following theorem is well known: There is a unique geodesic joining two points on a complete simply connected Riemannian manifold of nonpositive sectional curvature.

The main point of this paper is the following generalization.

Theorem. *Let N and B be minimal submanifolds of a Riemannian manifold M whose sectional curvature is nonpositive.* (If $\dim N = \dim M - 1$, it would suffice to know that M has nonpositive Ricci curvature.)
Suppose that:

a) *N is oriented and finite with oriented boundary $\partial N \subset B$.*
b) *B is a totally geodesic submanifold of M.*
c) *Each point p of N can be joined to B by a geodesic, which is perpendicular to B at the end-point, and varies smoothly with p.*

Conclusion: $N \subset B$.

The main tool is an integral-geometric inequality, which enables one to make various extensions of the main result, e.g., to the case where B is only a minimal submanifold of M, or where N is a manifold with singularities, e.g., a piece of an analytic subvariety of a Kähler manifold.

2. Proof of the theorem

Let M be a complete Riemannian manifold, and N and B submanifolds of M. (For notations not explained here, refer to [1] and [2].) Let exp: $T(M) \to M$ be the exponential map of the Riemannian structure, where $T(M)$ is the tangent bundle of M. Suppose there exists a vector field X on M such that:

a) For $p \in N$, $\exp(X(p)) \in B$.
b) The geodesic $t \to \exp(tX(p))$ is perpendicular to B at $t = 1$.

Let $\| \;\|$ denote the norm on tangent vectors associated with the inner product $\langle\, ,\rangle$ defining the Riemannian metric on M, $f(p) = \|X(p)\|^2$ for $p \in N$, and Δ^N be the Laplace-Beltrami operator, relative to the induced metric on N. Our

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