J. DIFFERENTIAL GEOMETRY
1 (1967) 257-267

REMARKS ON CLOSED MINIMAL SUBMANIFOLDS IN THE STANDARD RIEMANNIAN m-SPHERE

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Introduction

An r-dimensional minimal submanifold N^r of a Riemannian manifold M^m is a regularly imbedded sub-Riemannian manifold, which locally gives an extremal for r-dimensional volume, with fixed boundary, and therefore is the r-dimensional generalization of a geodesic curve [3, §52]. For this type of global geometric variational problem, it is naturally interesting to find the *closed* r-dimensional minimal submanifolds of a given Riemannian manifold M^m; in this respect, very few such submanifolds are known even in the simplest and nicest case in which M^m is the standard sphere S^m. Intuitively, closed minimal submanifolds of codimension one should be "fewer" than those with higher codimension, and hence "harder" to get. So far, all the known examples of closed minimal submanifolds of codimension one in S^m are "homogeneous" (see [6] for definition) and they are the extremal orbits of suitable isometry subgroups respectively.

In this short note, we shall begin with an observation that every *homogeneous* minimal submanifold N^r in S^m is *algebraic* in the following sense:

Consider S^m as the unit sphere in the euclidean space $\boldsymbol{R}^{m+1}$. Let $N^r \subseteq S^m$ be a closed minimal submanifold in S^m, O the origin of $\boldsymbol{R}^{m+1}$ and $\overrightarrow{Ox}$ the ray from O passing through a point $x \in \boldsymbol{R}^{m+1}$. Then it is clear that the cone

$$ON^r = \text{the union of } \overrightarrow{Ox} \text{ with } x \in N^r$$

is also *minimal in the euclidean space* $\boldsymbol{R}^{m+1}$ [1, §§6.15, 10.2]. We shall call N^r real algebraic if ON^r is a real algebraic cone.

It is well known that every complex algebraic cone is always minimal in $C^n = \boldsymbol{R}^{2n}$ [4, §§4.2, 4.10]. However, the codimension of a complex algebraic cone is at least two. Hence the problem of *real algebraic minimal cones of codimension one* seems to be more delicate than the complex case. Another motivation to investigate closed minimal subminifolds of codimension one in the standard sphere S^m is due to their closed relationship with the problem of interior regularity for non-parametric solutions to Plateau's problem in codimension 1 and higher dimensional generalizations of Bernstein's theorem [2].

Received June 10, 1967. Supported by National Science Foundation grant GP-5804.