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HOLOMORPHIC BISECTIONAL CURVATURE

SAMUEL I. GOLDBERG & SHOSHICHI KOBAYASHI

1. Introduction

Let M be a Kähler manifold of complex dimension n and R its Riemannian curvature tensor. At each point x of M, R is a quadrilinear mapping $T_x(M) \times T_x(M) \times T_x(M) \times T_x(M) \to \boldsymbol{R}$ with well known properties.

Let σ be a plane in $T_x(M)$, i.e., a real two dimensional subspace of $T_x(M)$. Choosing an orthonormal basis X, Y for σ, we define the sectional curvature $K(\sigma)$ of σ by

$$K(\sigma) = R(X, Y, X, Y) \ . \tag{1}$$

We shall occasionally write $K(X, Y)$ for $K(\sigma)$. The right hand side depends only on σ, not on the choice of an orthonormal basis X, Y. The sectional curvature K is a function defined on the Grassmann bundle of (two-) planes in the tangent spaces of M. A plane σ is said to be holomorphic if it is invariant by the (almost) complex structure tensor J. The set of J-invariant planes σ is a holomorphic bundle over M with fibre $P_{n-1}(C)$ (complex projective space of dimension $n - 1$). The restriction of the sectional curvature K to this complex projective bundle is called the holomorphic sectional curvature and will be denoted by H. In other words, $H(\sigma)$ is defined only when σ is invariant by J, and $H(\sigma) = K(\sigma)$. If X is a vector in σ we shall also write $H(X)$ for $H(\sigma)$.

Given two J-invariant planes σ and σ' in $T_x(M)$, we define the holomorphic bisectional curvature $H(\sigma, \sigma')$ by

$$H(\sigma, \sigma') = R(X, JX, Y, JY) \ , \tag{2}$$

where X is a unit vector in σ and Y a unit vector in σ'. It is a simple matter to verify that $R(X, JX, Y, JY)$ depends only on σ and σ'. Although the definition itself makes sense even for Hermitian holomorphic vector bundles (cf. Nakano [10]) as well as Hermitian manifolds we shall confine our considerations to the Kählerian case.

Since

$$H(\sigma, \sigma) = H(\sigma) \ , \tag{3}$$

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