J. DIFFERENTIAL GEOMETRY
1 (1967) 127–131

THE LEWY COUNTEREXAMPLE AND THE LOCAL EQUIVALENCE PROBLEM FOR G-STRUCTURES

V.W. GUILLEMIN & S. STERNBERG

1. Let G be a Lie subgroup of $GL(n)$. Let M_1 and M_2 be differential manifolds of dimension n (in this paper all data will be assumed to be C^∞), and let $\mathcal{F}_i$, $i = 1, 2$, be the principal frame bundle on M_i. A sub-bundle, P_i, of $\mathcal{F}_i$ with structure group G is called a G-structure on M_i. The G-structure on M_1 is said to be equivalent to the G-structure on M_2 if there exists a diffeomorphism $f : M_1 \to M_2$ such that the induced diffeomorphism $f^* : \mathcal{F}_1 \to \mathcal{F}_2$ carries P_1 into P_2.

It is usually difficult to decide when two G-structures are equivalent; however the problem is a little simpler if we suppose that one of the structures, say P_1, is locally transitive, and look only at the local problem. Then the following is a necessary condition for the two structures to be locally equivalent:

$*$ At every point $m_1 \in M_1$ and every point $m_2 \in M_2$ there exists a power series mapping ρ (in local coordinates with origins at m_1 and m_2) such that ρ formally effects a local equivalence between P_1 and P_2.

It might seem that (*) is not much of an improvement over the original problem; however, by techniques of homological algebra it can be converted into a much simpler statement about the vanishing of certain canonically defined tensors on P_2 (cf. [1], [2], [4]). The main problem therefore is to show that condition (*) is sufficient. This is known to be true in the following important cases:

1) G is of finite type.

2) The data are real analytic.

According to a recent result of Malgrange (unpublished) it is known to be true when G is elliptic. According to a result of the first author it is true when P_1 is flat. The purpose of this note is to show that condition (*) isn't always sufficient. In fact we will show that in certain cases the solution of the equivalence problem depends on the solution of a system of linear inhomogeneous partial differential equations resembling the Lewy counterexample [3]. These equations are determined, all the data in them are C^∞ and they have no solutions even in the weak (distribution) sense.

2. Let X_1, X_2 and X_3 be globally defined vector fields on R^3 satisfying the following commutation relations: $[X_1, X_2] = X_3, [X_1, X_3] = X_1, [X_2, X_3] = -X_2$. (Take for example the standard basis of so(3) and identify R^3 with a subset of SO(3) under the mapping exp: so(3) $\to$

Communicated June 26, 1967.