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MINIMAL IMMERSIONS OF SURFACES IN EUCLIDEAN SPHERES

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1. Introduction

The study of isolated singularities for minimal 3-dimensional varieties immersed in Euclidean n-space $\boldsymbol{E}^n$ requires as a first step a characterization of the tangent cone; the latter is the join of the origin 0 in $\boldsymbol{R}^n$ with a compact surface (the directrix) immersed in the unit Euclidean $(n-1)$-sphere S^{n-1} as a relatively minimal surface. Since comparatively little is known concerning such immersions, I propose to devote this as the first of a series of articles on the subject.

One may consider, for a start, a restricted type of singularity of a minimal 3-variety in $\boldsymbol{E}^n$, namely when this variety is topologically a manifold; in this case the directrix surface of the tangent cone is a 2-sphere immersed in S^{n-1} in a locally minimal way. This article is primarily devoted to minimal immersions of 2-spheres in Euclidean $(n-1)$-spheres. By this we mean immersions for which the total area is stationary with respect to variation, and minimal with respect to variation affecting sufficiently small portions of the surface at a time. Naturally, some of the conclusions developed here (through Lemma 5.3) apply to the minimal immersion of surfaces of positive genus as well; results pertaining to these will be collected elsewhere. In the case of minimal immersions of S^2 into the Euclidean sphere rS^{n-1} of radius r, the main result (Theorem 5.5) is that, if the image under such an immersion does not lie in any equatorial hyperplane section of rS^{n-1} then n is an odd integer and the area of the immersed S^2 is an integral multiple of $2\pi r^2$, at least equal to $\left(\dfrac{n^2-1}{8}\right)(4\pi r^2)$. There follow some discussion and examples to indicate why the above estimate is optimal.

2. Riemannian and Riemann surfaces

We denote by Σ an oriented surface, which, for the purposes of this article, may be assumed to be compact and either real analytic or differentiable. A differentiable Riemannian metric ds^2 on Σ together with the given orientation defines a covering of Σ by open domains with local (complex) isothermal parameters such as $w = u + iv$ $(i = \sqrt{-1})$ as well as its complex conjugate $\bar{w} = u - iv$. These parameters are defined up to a local holomorphic and holomorphically invertible transformation, and characterized by the following conditions.

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