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CURVATURE AND CHARACTERISTIC CLASSES OF COMPACT RIEMANNIAN MANIFOLDS

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In memory of Professor Vernon G. Grove

Introduction

During the past quarter century the development of the theory of fibre bundles has led to a new direction in differential geometry for studying relationships between curvatures and certain topological invariants such as characteristic classes of a compact Riemannian manifold. Along this direction the first and simplest result is the Gauss-Bonnet formula [2], [3], which expresses the Euler-Poincaré characteristic of a compact orientable Riemannian manifold of even dimension n as an integral of the n-th sectional curvature or the Lipschitz-Killing curvature times the element of area of the manifold. Later, Chern [5] obtained curvature conditions respectively for determining the sign of the Euler-Poincaré characteristic and for the vanishing of the Pontrjagin classes of a compact orientable Riemannian manifold. Recently, Thorpe [8] extended a special case of Chern's conditions by using higher order sectional curvatures, which are weaker invariants of the Riemannian structure than the usual sectional curvature. The purpose of this paper is to further extend the conditions of both Chern and Thorpe.

In §1, for a Riemannian manifold the equations of structure are given, and higher order sectional curvatures and related differential forms are defined. §2 contains the differential forms expressing, respectively, the Euler-Poincaré characteristic and the Pontrjagin classes of compact orientable Riemannian manifolds in the sense of de Rham's theorem. In §3, we first define some general curvature conditions, and then use them to extend the above mentioned results of Chern and Thorpe. The proofs of the main results (Theorems 3.1 and 3.2) of this paper are easily deduced from several lemmas.

1. Higher order sectional curvatures

Let M be a Riemannian manifold of dimension n (and class C^∞), and V_x, V_x^* respectively the spaces of tangent vectors and covectors at a point x of the manifold M. By taking an orthonormal basis in V_x and its dual basis in V_x^*, over a neighborhood U of the point x on the manifold M, we then have a family of orthonormal frames $xe_1 \cdots e_n$

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