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# CURVATURE AND THE EIGENVALUES OF THE LAPLACIAN

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## 1. Introduction

A famous formula of *H.* Weyl [19] states that if $D$ is a bounded region of $R^d$ with a piecewise smooth boundary $B$, and if $0 > \gamma_1 \geq \gamma_2 \geq \gamma_3 \geq$ etc. $\downarrow -\infty$ is the spectrum of the problem

$$\text{(1a)} \qquad \Delta f = (\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2) f = \gamma f \quad \text{in} \quad D,$$

$$\text{(1b)} \qquad f \in C^2(D) \ \cap \ C(\bar{D}),$$

$$\text{(1c)} \qquad f = 0 \quad \text{on} \quad B,$$

then

$$\text{(2)} \qquad -\gamma_n \ \sim \ C(d)(n/\text{vol } D)^{2/d}(n \uparrow \infty),$$

or, what is the same,

$$\text{(3)} \qquad Z \equiv \text{sp } e^{t\Delta} = \sum_{n\geq 1} \exp\,(\gamma_n t) \ \sim \ (4\pi t)^{-d/2} \times \text{vol } D \ (t \downarrow 0),$$

where $C(d) = 2\pi[d/2)!]^{d/2}$.

*Å*. Pleijel [13] and *M*. Kac [6] took up the matter of finding corrections to (3) for plane regions $D$ with a finite number of holes. The problem is to find how the spectrum of $\Delta$ reflects the shape of $D$. Kac puts things in the following amusing language : thinking of $D$ as a drum and $0 < -\gamma_1 < -\gamma_2 \leq$ etc. as its fundamental tones, *is it possible, just by listening with a perfect ear, to hear the shape of D*? Weyl's estimate (2) shows that you can hear *the area of D.* Kac proved that for $D$ bounded by a broken line $B$,

$$\text{(4a)} \qquad Z = \frac{\text{area } D}{4\pi t} - \frac{\text{length } B/4}{\sqrt{4\pi t}}$$
$$+ \text{the sum over the corners of } \frac{\pi^2 - \gamma^2}{24\pi\gamma} + \ o(1) \quad (t \downarrow 0),$$

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