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FOCAL SETS OF REGULAR MANIFOLDS M_{n-1} IN E_n

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1. Introduction

The object of this paper is to prove the following theorem and give it a proper setting in differential topology.

Theorem 1.1. *There exists a regular connected $(n-1)$-dimensional manifold M_{n-1} of class C^∞ in a euclidean space E_n such that the focal points of M_{n-1} are everywhere dense in E_n.*

In particular there exists a simple regular curve M_1 of class C^∞ in E_2 whose centers of curvature are everywhere dense in E_2. See §2.

The mainfold M_{n-1} of Theorem 1.1 is without any differentiable singularity in E_n and without self-intersection. However it cannot be compact by virtue of Theorem 1.2.

In Theorem 1.2 we refer to a subset of E_n of *J-content zero.* Given a positive constant e, such a set is characterized by the property that it is included in a finite number of n-rectangles whose total volume is less than e.

Theorem 1.2. *Let M_{n-1} be a regular manifold of class C^m, $m > 1$, in E_n, and let $\hat{M}_{n-1}$ and $\hat{E}_n$ be, respectively, relatively compact open subsets of M_{n-1} and E_n.*

Then the set of focal points of $\hat{M}_{n-1}$ in $\hat{E}_n$ has a J-content zero in E_n, implying that the set of focal points of $\hat{M}_{n-1}$ is nowhere dense in E_n.

Note. It is not affirmed that the set of focal points of $\hat{M}_{n-1}$ in E_n has J-content zero.

Theorem 1.2 admits an *extension* in which M_{n-1} is replaced by M_r where $0 < r < n$. Both Theorem 1.2 and its extension are provable by methods used by the author in his colloquium lectures in treating focal points of extremals "transverse" to a differentiable manifold. We shall establish Theorem 1.2 by non-variational methods later in this section. The extension of Theorem 1.2 can also be established by non-variational methods and this will be done in an introduction to critical point theory in global analysis and differential topology now being written.

Theorem 1.2 *implies, but is not implied by, the theorem that the set of focal points of the manifold M_{n-1} in Theorem* 1.2 *has a Lebesgue measure zero in E_n.*

We shall recall some essential definitions.

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