J. DIFFERENTIAL GEOMETRY
53 (1999) 183-202

THE AFFINE SOBOLEV INEQUALITY

GAOYONG ZHANG

1. Introduction

The Sobolev inequality is one of the fundamental inequalities connecting analysis and geometry. The literature related to it is vast (see, for example, [1], [5], [7], [3], [6], [11], [12], [19], [22], [20], [21], [23], [25], [27], [28], [37], and [45]). In this paper, a new inequality that is stronger than the Sobolev inequality is presented. A remarkable feature of the new inequality is that it is independent of the norm chosen for the ambient Euclidean space.

The Sobolev inequality in the Euclidean space $\mathbb{R}^n$ states that for any C^1 function $f(x)$ with compact support there is

$$\int_{\mathbb{R}^n} |\nabla f| \, dx \geq n\omega_n^{1/n} \|f\|_{\frac{n}{n-1}}, \tag{1.1}$$

where $|\nabla f|$ is the Euclidean norm of the gradient of f, $\|f\|_p$ is the usual L_p norm of f in $\mathbb{R}^n$, and ω_n is the volume enclosed by the unit sphere S^{n-1} in $\mathbb{R}^n$. The best constant in the inequality is attained at the characteristic functions of balls.

It is known that the sharp Sobolev inequality (1.1) is equivalent to the classical isoperimetric inequality (see, for instance, [2], [8], [13], [14], [33], [35], [40], and [41]). We prove an affine Sobolev inequality which is stronger than (1.1). This inequality is proved by using a generalization of the Petty projection inequality to compact sets that is established in this paper (see [30], [31], [16], [26], [38] and [42] for the classical Petty projection inequality of convex bodies).

Received February 2, 2000. Research supported, in part, by NSF Grant DMS-9803261