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CRITICAL SETS OF SOLUTIONS TO ELLIPTIC EQUATIONS

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Abstract

Let $u \not\equiv$ const. satisfy an elliptic equation $\mathcal{L}_0 u \equiv \sum a_{ij} D_{ij} u + \sum b_j D_j u = 0$ with smooth coefficients in a domain in $\mathbf{R}^n$. It is shown that the critical set $|\nabla u|^{-1}\{0\}$ has locally finite $(n-2)$-dimensional Hausdorff measure. This implies in particular that for a solution $u \not\equiv 0$ of $(\mathcal{L}_0 + c)u = 0$, with $c \in C^\infty$, the singular set $u^{-1}\{0\} \cap |\nabla u|^{-1}\{0\}$ has locally finite $(n-2)$-dimensional Hausdorff measure.

1. Introduction and main results

Let Ω be a domain in $\mathbf{R}^n$, $n \geq 3$, and let $u \not\equiv 0$ be a real-valued classical solution of the elliptic partial differential equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_j D_j u + cu = 0 \quad \text{in } \Omega, \tag{1.1}$$

where the real-valued coefficients a_{ij}, b_j, c are C^∞ functions in Ω. We call

$$\Sigma(u) = |\nabla u|^{-1}\{0\} \text{ and } \Sigma_0(u) = \Sigma(u) \cap u^{-1}\{0\}$$

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