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COLLAPSED MANIFOLDS WITH PINCHED POSITIVE SECTIONAL CURVATURE

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Abstract

Let M^n be a manifold of sectional curvature, $0 < \delta \le K_{M^n} \le 1$, let X be an Alexandrov space of curvature ≥ -1. Suppose the Gromov-Hausdorff distance of M^n and X is less than $\epsilon(n,\delta) > 0$. Our main results are: (A) If X has the lowest possible dimension, $\frac{n-1}{2}$, then a covering space of M^n of order $\le \frac{n+1}{2}$ is diffeomorphic to a lens space, $S^n/\mathbb{Z}_q$, such that $0 < c(n,\delta)[vol(M^n)]^{-1} \le q \le vol(S^n_\delta)[vol(M^n)]^{-1}$, where S^n_δ is the sphere of constant curvature δ. (B) If X has nonempty boundary, then a covering space of M^n of order $\le \frac{n+1}{2}$ is diffeomorphic to a lens space, provided ϵ depends also on the Hausdorff measure of X.

0. Introduction

Let d_{GH} denote the Gromov-Hausdorff distance between two metric spaces, cf. [20]. Gromov's theory of almost flat manifolds asserts that a compact manifold, M^n, whose finite normal covering of order $\le i(n)$ (the Margulis constant) is diffeomorphic to a compact nilpotent manifold, N/Γ, if and only if M^n admits a metric with sectional curvature $|K_{M^n}| \le 1$ and $d_{GH}(M^n, pt) < \epsilon(n)$, a small constant depending only on n, see [6], [19] and [36].

In this paper, one of the problems we shall be concerned with is to characterize a compact manifold, M^n, which admits a metric with $0 < \delta \le K_{M^n} \le 1$ such that $d_{GH}(M^n, X)$ is sufficiently small depending only on n and δ, where X is an *Alexandrov space* of the lowest dimension with n and δ fixed, cf. [4] (see Theorem 0.4). Since the diameter of M^n

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