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THE WEYL UPPER BOUND ON THE DISCRETE SPECTRUM OF LOCALLY SYMMETRIC SPACES

LIZHEN JI

1. Introduction

1.1. Let G be a reductive Lie group with finitely many connected components, and Γ a cofinite volume discrete subgroup of G. Let $K \subset G$ be a maximal compact subgroup, and $X = G/K$ be the associated symmetric space, which is the product of a symmetric space of noncompact type and a possible Euclidean space. Then $\Gamma\backslash X$ is a locally symmetric space of finite volume. For simplicity, we assume, unless otherwise specified, that there exists a reductive algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ satisfying the conditions in [18, p. 1] such that $G = \mathbf{G}(\mathbb{R})$, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup.

Any finite dimensional unitary representation σ of K defines a homogeneous bundle $\tilde{E}_\sigma$ on X and hence a locally homogeneous bundle E_σ on $\Gamma\backslash X$. The bundle E_σ admits a locally invariant connection $\bigtriangledown$ which is the push forward of the invariant connection on the homogeneous bundle $\tilde{E}_\sigma$. The connection $\bigtriangledown$ defines a quadratic form D on sections of E_σ: For any $f \in C_0^\infty(\Gamma\backslash X, \sigma)$,

$$D(f) = \int_{\Gamma\backslash X} |\bigtriangledown f(x)|^2 dx.$$

This quadratic form D defines an elliptic operator Δ on $L^2(\Gamma\backslash X, \sigma)$, called the Laplace operator, where $L^2(\Gamma\backslash X, \sigma)$ denotes the space of L^2-sections of E_σ. If σ is irreducible, Δ is equal to a shift of the restriction of

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