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WEIL-PETERSON CONVEXITY OF THE ENERGY FUNCTIONAL ON CLASSICAL AND UNIVERSAL TEICHMÜLLER SPACES

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1. Introduction

1.1. Weil-Petersson convexity and classical Teichmüller space. Suppose that we have a smooth compact Riemanninan mainfold M^n of dimension n with a metric g, and a compact surface N^2 with a hyperbolic metric G. We assume that both M and N have no boundary. Let $\{x^i\}$ be a local coordinate for M^n, and $\{y^\alpha\}$ a local coordinate for N^2.

The following statements follow from the results of Eells, Sampson [3] and Hartman [8] as well as Al'bers [1].

Theorem. *Given a continuous map $\phi : M^n \to N^2$, there is a smooth harmonic map $u : M^n \to N^2$ homotopic to ϕ, and u is unique in the homotopy class, unless the image of the map is a point or a closed geodesic in N.*

Eells and Lemiare [4] have shown that as long as harmonic maps exist and are uniquely determined, when the image metric is varied smoothly along a curve G^t (with $G^0 = G$), the harmonic maps $u_t : (M, g) \to (N, G^t)$ vary smoothly without changing homotopy type in the parameter t for sufficiently small t. (In order to ensure the existence and uniqueness of u_t for all t, we require the negativity of the sectional curvature of G^t.) In particular, the energy functional;

$$\mathcal{E}(t) = \frac{1}{2}\int_{M^n} G^t_{\alpha\beta}(u_t) g^{ij} \frac{\partial u_t^\alpha}{\partial x^i}\frac{\partial u_t^\beta}{\partial x^j} d\mu_g$$

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