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SHORTENING COMPLETE PLANE CURVES

KAI-SENG CHOU & XI-PING ZHU

Abstract

It is shown that the curve shortening problem for a complete, properly embedded curve has a solution for all time provided the initial curve divides the plane into two regions of infinite area.

Introduction

In the curve shortening problem one studies the evolution of a plane curve γ_0 under the equation

$$\frac{\partial \gamma}{\partial t} = k\nu \tag{0.1}$$

where ν is a choice of unit normal and k is the curvature of $\gamma(\cdot, t)$ with respect to ν. When γ_0 is closed and convex, $\gamma(\cdot, t)$ shrinks under (0.1). In fact, it was proved by Gage and Hamilton [12] that a convex closed curve shrinks to a round point in finite time. Subsequently Grayson [13] showed that a closed embedded curve evolves into a convex curve before it shrinks to a point. Thus the curve shortening problem for closed embedded curves is completely solved. In this paper we study (0.1) for complete, noncompact embedded curves. Very few results in this direction are known. In [9] and [10] Ecker and Huisken studied the mean curvature flow for hypersurfaces which are either local or global graphs. Among other things they proved that the mean curvature flow for entire graphs exists for all time. Problem (0.1) is the mean curvature flow for curves. Their result asserts that for a complete initial curve which is a graph over a straight line, (0.1) has a solution for all $t \geq 0$.

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