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A HOLOMORPHIC CASSON INVARIANT FOR CALABI-YAU 3-FOLDS, AND BUNDLES ON $K3$ FIBRATIONS

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Abstract

We briefly review the formal picture in which a Calabi-Yau n-fold is the complex analogue of an oriented real n-manifold, and a Fano with a fixed smooth anticanonical divisor is the analogue of a manifold with boundary, motivating a holomorphic Casson invariant counting bundles on a Calabi-Yau 3-fold. We develop the deformation theory necessary to obtain the virtual moduli cycles of [31], [7] in moduli spaces of stable sheaves whose higher obstruction groups vanish. This gives, for instance, virtual moduli cycles in Hilbert schemes of curves in $\mathbb{P}^3$, and Donaldson– and Gromov-Witten– like invariants of Fano 3-folds. It also allows us to define the holomorphic Casson invariant of a Calabi-Yau 3-fold X, prove it is deformation invariant, and compute it explicitly in some examples. Then we calculate moduli spaces of sheaves on a general $K3$ fibration X, enabling us to compute the invariant for some ranks and Chern classes, and equate it to Gromov-Witten invariants of the "Mukai-dual" 3-fold for others. As an example the invariant is shown to distinguish Gross' diffeomorphic 3-folds. Finally the Mukai-dual 3-fold is shown to be Calabi-Yau and its cohomology is related to that of X.

1. Introduction

This paper is a continuation of the ideas presented in [12], [40]. There a formal picture was outlined in which the complex analogue of a real *oriented* n-manifold is a *Calabi-Yau* n-fold with a fixed holomorphic n-form playing the role of a "complex orientation", while a Fano with fixed smooth anticanonical divisor is the analogue of a manifold with boundary; the boundary being the (Calabi-Yau) divisor. I have since discovered this picture was known and used in low dimensions by the Yale school of Frenkel, Khesin, Todorov and others (see for instance [13],

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