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NONLINEAR EVOLUTION BY MEAN CURVATURE AND ISOPERIMETRIC INEQUALITIES

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Abstract

Evolving smooth, compact hypersurfaces in $\mathbb{R}^{n+1}$ with normal speed equal to a positive power k of the mean curvature improves a certain 'isoperimetric difference' for $k \geqslant n-1$. As singularities may develop before the volume goes to zero, we develop a weak level-set formulation for such flows and show that the above monotonicity is still valid. This proves the isoperimetric inequality for $n \leqslant 7$. Extending this to complete, simply connected 3-dimensional manifolds with nonpositive sectional curvature, we give a new proof for the Euclidean isoperimetric inequality on such manifolds.

1. Introduction

Let M^n be a smooth n-dimensional compact manifold without boundary and $F_0 : M^n \to N^{n+1}$ a smooth embedding into an n+1-dimensional Riemannian manifold $(N^{n+1}, \bar{g})$. We assume further that $F_0(M)$ has positive mean curvature in N^{n+1}. Starting from such an initial hypersurface there exists, at least for a short time interval $[0, T)$, an evolution $F(\cdot, t) : M^n \times [0, T) \to N^{n+1}$, which satisfies

$$(\star) \qquad \begin{cases} F(\cdot, 0) = F_0(\cdot) \\ \dfrac{dF}{dt}(\cdot, t) = -H^k(\cdot, t)\nu(\cdot, t) \end{cases}$$

where $k \geqslant 1$, H is the mean curvature and ν is the outer unit normal, such that $-H\nu = \mathbf{H}$ is the mean curvature vector. Let $A(t)$ denote the area of such an evolving hypersurface, $V(t)$ the enclosed volume, and c_{n+1} the Euclidean isoperimetric constant. We aim to exploit the following fact, to which G. Huisken has drawn our attention: the 'isoperimetric difference'

$$(1) \qquad A(t)^{\frac{n+1}{n}} - c_{n+1} V(t)$$

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