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PROOF OF THE ANGULAR MOMENTUM-MASS INEQUALITY FOR AXISYMMETRIC BLACK HOLES

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Abstract

We prove that an extreme Kerr initial data set is a unique absolute minimum of the total mass in a (physically relevant) class of vacuum, maximal, asymptotically flat, axisymmetric data for Einstein equations with fixed angular momentum. These data represent non-stationary, axially symmetric black holes.

As a consequence, we obtain that any data in this class satisfy the inequality $\sqrt{J} \leq m$, where m and J are the total mass and angular momentum of spacetime.

1. Introduction

An *initial data set* for the Einstein vacuum equations is given by a triple (S, h_{ij}, K_{ij}) where S is a connected 3-dimensional manifold, h_{ij} a (positive definite) Riemannian metric, and K_{ij} a symmetric tensor field on S, such that the vacuum constraint equations

$$D_j K^{ij} - D^i K = 0, \tag{1}$$

$$R - K_{ij}K^{ij} + K^2 = 0, \tag{2}$$

are satisfied on S. D and R are the Levi-Civita connection and the Ricci scalar associated with h_{ij}, and $K = K_{ij}h^{ij}$. In these equations the indices are moved with the metric h_{ij} and its inverse h^{ij}.

The manifold S is called *Euclidean at infinity* if there exists a compact subset $\mathcal{K}$ of S such that $S \setminus \mathcal{K}$ is the disjoint union of a finite number of open sets U_k, and each U_k is isometric to the exterior of a ball in $\mathbb{R}^3$. Each open set U_k is called an *end* of S. Consider one end U and the canonical coordinates x^i in $\mathbb{R}^3$, which contains the exterior of the ball to which U is diffeomorphic. Set $r = \left(\sum (x^i)^2\right)^{1/2}$. An initial data set is called *asymptotically flat* if S is Euclidean at infinity, the metric h_{ij} tends to the euclidean metric, and K_{ij} tends to zero as $r \to \infty$ in an appropriate way. These fall off conditions (see [**2**], [**13**] for the optimal

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