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THE LENGTH OF A SHORTEST GEODESIC LOOP AT A POINT

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Abstract

In this paper we prove that given a point $p \in M^n$, where M^n is a closed Riemannian manifold of dimension n, the length of a shortest geodesic loop $l_p(M^n)$ at this point is bounded above by $2nd$, where d is the diameter of M^n. Moreover, we show that on a closed simply connected Riemannian manifold M^n with a non-trivial second homotopy group there either exist at least three geodesic loops of length less than or equal to $2d$ at each point of M^n, or the length of a shortest closed geodesic on M^n is bounded from above by 4d.

Introduction and main results

Let M^n be a closed Riemannian manifold of dimension n. In 1983, M. Gromov asked whether one can bound above the length of a shortest closed geodesic $l(M^n)$ on M^n by $c(n)\mathrm{vol}\,(M^n)^{\frac{1}{n}}$, where $\mathrm{vol}\,(M^n)$ is the volume of M^n and $c(n)$ is a constant that depends on the dimension of M^n only. A similar question can be asked about the relationship between $l(M^n)$ and the diameter d of a manifold. The fact that on each manifold there exists a closed geodesic was shown by L. Lusternik and A. Fet. A similar argument shows that there exists a geodesic loop at each point of a closed Riemannian manifold. So, one can also ask if there exists a constant $k(n)$ such that for each point $p \in M^n$, the length of a shortest geodesic loop $l_p(M^n)$ at this point is bounded above by $k(n)d$. Note that, although it is quite easy to see that $l_p(M^n) \leq 2d$ in the case of a closed Riemannian manifold that is not simply connected, this is not true in general, as it was recently shown by F. Balacheff, C.B. Croke, and M. Katz in [**BlCK**].

Note also, that for no constant $C(n)$ we can bound above $l_p(M^n)$ by $C(n)\mathrm{vol}\,(M^n)^{\frac{1}{n}}$ for every $p \in M^n$. For example, consider a prolate ellipsoid E that is an ellipsoid generated by an ellipse rotated around its major axis. Let us denote its polar radius by R. Let $p \in E$ be the north pole of E. Then all geodesics and, thus, geodesic loops passing through

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